PARAMETRIZATION OF LINEAR COUPLED MOTION IN PERIODIC SYSTEMS

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Summary

The Courant-Snyder parametrization for one-dimensional linear motion in periodic systems is generalized to two-dimensional coupled linear motions. The 10-parameter 4x4 symplectic transfer matrix across a period is expressed in two normal-mode invariant phase-advances, four normal-mode periodic "amplitude" functions, and four periodic functions which reflect the strength and structure of the coupling. These parametric functions are also given by differential equations containing the periodic force coefficients appearing in the Hamiltonian. Two bilinear invariants can be constructed. With given horizontal and vertical aperture limitations, these invariants give the four-dimensional phase-space acceptance volume and the propagation of the horizontal and vertical emittances of the beam.

Introduction

Over a year ago, as the NAL main accelerator was being brought into operation, it was clear that a substantial coupling existed between radial and vertical betatron oscillations. Last summer, it became important to take steps to compensate the coupling, for not only was the interpretation of low-field betatron oscillation phenomena obscured by the interplay between these two degrees of freedom but also a serious degradation of the vertical emittance of the extracted beam could be anticipated. We were thus led to a reexamination of two-dimensional linear coupling. At the same time, a parallel experimental effort was initiated to reduce the coupling effects. By the end of summer, a combination of trim skew quadrupoles and main quadrupole rotations had reduced the coupling effects to proportions tolerable for the present. The pressure of other more critical matters caused us to lay aside our treatment of the linear coupling problem, at the point where a formal solution had been found in a fashion which is an extension of the familiar Courant and Snyder[1] formulation of the one-dimensional case. We present that formal solution below, in which we introduce periodic parameters, in addition to those of Courant and Snyder, to reflect the coupling forces. What is lacking at this stage is an intuitive appreciation of the significance of these additional parameters akin to that which has been developed for the Courant-Snyder β's and α's over the past fifteen years or so.

Notation

When representing arrays, letters preceding "o" in the alphabet will be used for 2x2 matrices and letters after "o" will be used for 4x4 matrices and 4-element column vectors. Three exceptions to this rule will be made: the identify matrix is I regardless of size, the 4x4 Hamiltonian matrix will be denote by H, and since the context will always prevent misunderstanding, S will stand for either the 2x2 or 4x4 version of the unit symplectic matrix

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{1}$$

A "prime" will imply differentiation with respect to the independent variable z. The transpose of a matrix A is $\tilde{A}$, the trace is TrA, and the determinant is $|A|$.

Statement of the Problem

We consider a two-dimensional system described by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(p_x^2+p_y^2)+L(p_x y-p_y x)+\frac{1}{2}Fx^2+Kxy+\frac{1}{2}Gy^2 \tag{2}$$

where x and y are orthogonal coordinates, p_x and p_y are their conjugate momenta, and L, F, K, and G are periodic in z with a period of C. This form of the Hamiltonian may describe coupled betatron oscillations in a synchrotron, in which case K and L would arise from skew quadrupole and solenoidal fields, respectively. However, any bilinear Hamiltonian can be brought to this form by a suitable canonical transformation.

In matrix notation, (2) is

$$\mathcal{H} = \frac{1}{2}\tilde{X}HX \tag{3}$$

with

$$X \equiv \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}; \qquad H \equiv \begin{pmatrix} F & 0 & K & -L \\ 0 & 1 & L & 0 \\ K & L & G & 0 \\ -L & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

The canonical equations may be written

$$X' = QX \tag{5}$$

where

$$Q = SH = \begin{pmatrix} 0 & 1 & L & 0 \\ -F & 0 & K & L \\ -L & 0 & 0 & 1 \\ -K & -L & -G & 0 \end{pmatrix} \tag{6}$$

and S is the matrix defined in (1).

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We will find a canonical transformation R to new coordinates V of the form

$$V \equiv \begin{pmatrix} u \\ p_u \\ v \\ p_v \end{pmatrix} = R^{-1}X \tag{7}$$

such that the motions described by u and v are decoupled; solutions for each can then be written in Courant-Snyder phase-amplitude form containing periodic parameters analogous to their β's and α's. The transformation R will contain additional periodic parameters, reflecting the coupling terms in the Hamiltonian. The periodic parameters will be related to the single-turn matrix in a fasion similar to the one-dimensional case. Thus, we will have obtained a formal solution to the system (5). We will then exhibit two bilinear invariants satisfied by V, and so, through R, by X as well.

Selection of Parameters

A 2nx2n matrix T which satisfies the condition

$$\tilde{T}ST = S \tag{8}$$

where S is the 2nx2n generalization of (1) is called "symplectic." The Jacobian matrix of a canonical transformation has this property.[2] For linear systems, the transfer matrix $T(z_2,z_1)$ relating the state of the motion at z_2 to that at z_1 according to

$$X(z_2) = T(z_2,z_1)X(z_1) \tag{9}$$

is given by the Jacobian matrix of the canonical transformation from $X(z_1)$ to $X(z_2)$. $T(z_2,z_1)$ is therefore symplectic. The condition (8) gives $n(2n-1)$ relations among the $(2n)^2$ elements of T, so the number of independent parameters is $n(2n+1)$. In the case of two-dimensional motion, T will have 10 independent elements insofar as algebraic relationships are concerned.

For our periodic system, the "single-turn" transfer matrix defined by

$$T(z) \equiv T(z+C,z) \tag{10}$$

is also periodic. So T(z) can be expressed in terms of 10 periodic parameters, and this we will proceed to do.

Teng has presented a variety of ways in which symplectic matrices can be parametrized so that only quantities that are independent after taking the symplectic condition into account appear.[3] We will use his "symplectic rotation" form, which for two-dimensional motion is

$$T(z) = \begin{pmatrix} I\cos\phi & D^{-1}\sin\phi \\ -D\sin\phi & I\cos\phi \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I\cos\phi & -D^{-1}\sin\phi \\ D\sin\phi & I\cos\phi \end{pmatrix} \tag{11}$$

or

$$T(z) = RUR^{-1}. \tag{12}$$

In (11), A, B, and D are 2x2 unimodular (symplectic) matrices each of which requires 3 parameters; the tenth is the angle ϕ. Equation (12) serves as the definition of the 4x4 matrices R and U by association with (11). The notation of (12) anticipates the result that the matrix R therein will be used for the matrix of the same name in (7).

Replacing X by RV in (9) shows that U(z) is the single-turn matrix for V. It is natural then to parametrize the constituents of U--that is, A and B--in Courant-Snyder form:

$$\begin{aligned} A &= I\cos\mu_1 + J_1\sin\mu_1; \quad J_1 \equiv \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix} \\ B &= I\cos\mu_2 + J_2\sin\mu_2; \quad J_2 \equiv \begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} \end{aligned} \tag{13}$$

The phase advances μ_1 and μ_2 relate in the usual way to the eigenvalues of A and B; since T and U are related by a similarity transformation, T and U have the same eigenvalues.

Equation (13) specifies 6 of the periodic parameters, and the rest are ϕ and 3 describing the unimodular matrix D. The expressions below in which the elements of D appear are not significantly simplified by writing D in terms of 3 independent quantities so we will take

$$D = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{14}$$

bearing in mind that ad-bc = 1.

The parameters may be found in terms of the elements of T(z) by inversion of (11). Define 2x2 matrices M, m, N, and n by

$$T = \begin{pmatrix} M & n \\ m & N \end{pmatrix} . \tag{15}$$

Then (11) becomes

$$\begin{aligned} M &= A\cos^2\phi + D^{-1}BD\sin^2\phi \\ N &= B\cos^2\phi + DAD^{-1}\sin^2\phi \\ m &= -(DA-BD)\sin\phi\cos\phi \\ n &= -(AD^{-1}-D^{-1}B)\sin\phi\cos\phi \end{aligned} \tag{16}$$

After some matrix manipulations which need not be detailed here, we have

$$\begin{aligned} \cos\mu_1-\cos\mu_2 &= \frac{1}{2}\,\mathrm{Tr}(M-N)\left\{1 + \frac{2|m|+\mathrm{Tr}(nm)}{\left[\frac{1}{2}\mathrm{Tr}(M-N)\right]^2}\right\}^{1/2} \\ \cos 2\phi &= \frac{\frac{1}{2}\mathrm{Tr}(M-N)}{\cos\mu_1-\cos\mu_2} \\ D &= -\frac{m + S\tilde{n}\tilde{S}}{(\cos\mu_1-\cos\mu_2)\sin 2\phi} \\ A &= M-D^{-1}m\tan\phi; \qquad B = N+Dn\tan\phi \end{aligned} \tag{17}$$

where use has been made of the symplectic character of A, B, and D. In arriving at (17), an ambiguity in sign has been removed by restricting ϕ to the range $-\pi/4 \le \phi \le \pi/4$. A relative sign between D and $\sin 2\phi$ remains to be specified. Pursuing the resemblance between (11) and an ordinary rotation of coordinates, we will fix the sign of $\sin 2\phi$ by requiring that the trace of D be nonnegative.

Now we can show that use of R, as defined in (11) and (12), in the transformation from X to V according to X = RV, produces the desired decoupling of u and v. Transformation of the canonical equation (5) yields

$$V' = PV; \qquad P \equiv R^{-1}QR - R^{-1}R'. \tag{18}$$

But we are in a position to exhibit P. From (17), we know the elements of R in terms of those of T. R', through (17), is known in terms of T', which is given by

$$T' = QT - TQ \equiv [Q,T]. \tag{19}$$

The result is

$$P = \begin{bmatrix} -\frac{1}{2}[(a-d)L+bK]\tan\phi & 1-bL\tan\phi & 0 & 0 \\ -F-(cL-aK)\tan\phi & \frac{1}{2}[(a-d)L+bK]\tan\phi & 0 & 0 \\ 0 & 0 & \frac{1}{2}[(a-d)L-bK]\tan\phi & 1+bL\tan\phi \\ 0 & 0 & -G+(cL-dK)\tan\phi & \frac{1}{2}[(d-a)L+bK]\tan\phi \end{bmatrix} \tag{20}$$

Thus P is diagonal in 2x2 matrices; u and v are "normal" coordinates.

We note in passing that, in terms of our parametrization, a generating function for the canonical transformation from X to V is

$$W(x,y,p_u,p_v,z) = \frac{1}{1+bc\sin^2\phi} \tag{21}$$

$$\Big\{-ac\sin^2\phi x^2 - c\sin\phi\cos\phi xy + cd\sin^2\phi y^2$$

$$+ \cos\phi x p_u + a\sin\phi x p_v - d\sin\phi y p_u$$

$$+ \cos\phi y p_v - bd\sin^2\phi p_u^2 + b\sin\phi\cos\phi p_u p_v$$

$$+ ab\sin^2\phi p_v^2.\Big\}$$

We will not make explicit use of (21) in the sequel.

Differential Equations Satisfied by the Periodic Parameters

The dynamics of the system imposes differential relationships among the periodic parameters; we list them here. The derivatives of the parameters of R were found in the steps leading to the expression for the P matrix (20). They are

$$\phi' = \frac{1}{2}(a+d)L - \frac{1}{2}bK$$

$$a' = c+bF - \{[a(a+d)-2]L-abK\}\cot 2\phi$$

$$b' = d-a - \{b(a+d)L-b^2K\}\cot 2\phi \tag{22}$$

$$c' = -aG+dF - \{c(a+d)L-(1+ad)K\}\cot 2\phi$$

$$d' = -c-bG - \{[d(a+d)-2]L-bdK\}\cot 2\phi$$

The derivatives of the elements of U are obtained from U' = [P,U]:

$$\alpha_1' = -\gamma_1+\beta_1F + (bL\gamma_1-aK\beta_1+cL\beta_1)\tan\phi$$

$$\beta_1' = -2\alpha_1 + [(a-d)L\beta_1+bK\beta_1-2bL\alpha_1]\tan\phi$$

$$\gamma_1' = 2F\alpha_1 + [-2aK\alpha_1+2cL\alpha_1+(a-d)L\gamma_1+bK\gamma_1]\tan\phi$$

$$\alpha_2' = -\gamma_2+\beta_2G - [bL\gamma_2+cL\beta_2-dK\beta_2]\tan\phi \tag{23}$$

$$\beta_2' = -2\alpha_2 + [(a-d)L\beta_2-bK\beta_2-2bL\alpha_2]\tan\phi$$

$$\gamma_2' = 2G\alpha_2 - [2cL\alpha_2-2dK\alpha_2+(a-d)L\gamma_2-bK\gamma_2]\tan\phi$$

Solution of the Equations of Motion

Hamilton's canonical equations for the transformed variables are

$$V' = PV \tag{24}$$

with P given by (20). Since one has a very good idea how the solutions should look, (24) is easy to solve. Using the differential equations of the preceding section, one may verify that

$$u = (W_1\beta_1)^{\frac{1}{2}}\cos\psi_1$$

$$p_u = -\left(\frac{W_1}{\beta_1}\right)^{\frac{1}{2}}[\sin\psi_1+\alpha_1\cos\psi_1] \tag{25}$$

$$v = (W_2\beta_2)^{\frac{1}{2}}\cos\psi_2$$

$$p_v = -\left(\frac{W_2}{\beta_2}\right)^{\frac{1}{2}}[\sin\psi_2+\alpha_2\cos\psi_2]$$

are solutions provided that we define

$$\psi_1 = \int \frac{(1-bL\tan\phi)}{\beta_1}\,dz-\delta_1 \tag{26}$$

$$\psi_2 = \int \frac{(1+bL\tan\phi)}{\beta_2}\,dz-\delta_2$$

As usual, the four quantities W_1, W_2, δ_1, and δ_2 are the "constants of integration." In particular, W_1 and W_2 are the bilinear invariants

$$W_1 = \tilde{V}\tilde{S}\begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix} V; \qquad W_2 = \tilde{V}\tilde{S}\begin{pmatrix} 0 & 0 \\ 0 & J_2 \end{pmatrix} V \tag{27}$$

as may be seen by eliminating ψ_1 and ψ_2 between the appropriate pairs in (25).

The matrix $U(z_2,z_1)$, which conveys V from z_1 to z_2, when expressed in the periodic parameters is precisely of the form that we would expect from Courant and Snyder. We need not reproduce it here. The phases between z_1 and z_2, of course, must be consistent with (26), i.e.

$$\psi_i(z_2,z_1) = \psi_i(z_2) - \psi_i(z_1); \qquad i = 1,2. \tag{28}$$

Finally, the matrix $T(z_2,z_1)$ defined by (9) is

$$T(z_2,z_1) = R(z_2)U(z_2,z_1)R^{-1}(z_1). \tag{29}$$

With (29), we conclude the formal solution of the problem.

Comments on the Invariants

The two bilinear invariants (27), when transformed to x,y coordinates, are

$$W_1 = \tilde{X}\tilde{S}R\begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix} R^{-1}X; \quad W_2 = \tilde{X}\tilde{S}R\begin{pmatrix} 0 & 0 \\ 0 & J_2 \end{pmatrix} R^{-1}X \tag{30}$$

A particle projected with given initial conditions X, hence particular values of W_1 and W_2--E_1 and E_2 say--will move on a 2-surface characterized by $W_1 = E_1$ and $W_2 = E_2$. Neither $W_1 = E_1$ nor $W_2 = E_2$ individually describes a bounded region in 4-space, since two of the eigenvalues of each form are zero. Provided the motion is stable--i.e., if μ_1 and μ_2 are real, it is easy to show that the other two eigenvalues of each invariant form are positive.

In contrast to the one-dimensional case, the projection of the motion at fixed z onto either the x, x' plane or the y, y' plane is not in general an ellipse. As an example, consider the projection onto x, x'. Using solutions of the form (25) and the transformation R, we have

$$\begin{aligned} x &= (W_1\beta_1)^{\frac{1}{2}}\cos\phi\cos\psi_1 + d(W_2\beta_2)^{\frac{1}{2}}\sin\phi\cos\psi_2 \\ x' &= -\left(\frac{W_1}{\beta_1}\right)^{\frac{1}{2}}\cos\phi\sin\psi_1 - a\left(\frac{W_2}{\beta_2}\right)^{\frac{1}{2}}\sin\phi\sin\psi_2 \end{aligned} \tag{31}$$

where, to simplify the algebra, we have taken $\alpha_1 = \alpha_2 = b = c = 0$. (It is not hard to find systems for which these four parameters vanish at some z, but space does not permit us to go into detail on this point.) At fixed z, the maximum in x' for each x will occur when ψ_1 and ψ_2 are related by

$$\tan\psi_2 = \frac{a\beta_1}{d\beta_2}\tan\psi_1. \tag{32}$$

In terms of ψ_1, the boundary of the motion in the x, x' plane is then

$$\begin{aligned} x &= (W_1\beta_1)^{\frac{1}{2}}\cos\phi\cos\psi_1 \left[1 + \left(\frac{W_2\beta_2}{W_1\beta_1}\right)^{\frac{1}{2}} \frac{d^2\beta_2\tan\phi}{[(a\beta_1\sin\psi_1)^2 + (d\beta_2\cos\psi_1)^2]^{\frac{1}{2}}}\right] \\ x' &= -\left(\frac{W_1}{\beta_1}\right)^{\frac{1}{2}}\cos\phi\sin\psi_1 \left[1 + \left(\frac{W_2\beta_1}{W_1\beta_2}\right)^{\frac{1}{2}} \frac{a^2\beta_1\tan\phi}{[(a\beta_1\sin\psi_1)^2 + (d\beta_2\cos\psi_1)^2]^{\frac{1}{2}}}\right] \end{aligned} \tag{33}$$

If $\phi = 0$, (33) reduces to the familiar form. However, for non-zero ϕ, the coupling terms result in a deformation of the ellipse.

References

1. E.D. Courant and H.S. Snyder, Annals of Physics 3, 1 (1958).
2. See, for example, Whitaker's Analytical Dynamics.
3. L.C. Teng, NAL Report FN-229, May 3, 1971.