SKEW COLLIMATOR IMPEDANCE

$$Z = \frac{j}{\beta I \Delta x} \int_0^{2\pi R} (E + vB)\, ds$$ 1D impedance $\beta = v/c$

$$\langle \underline{F}_\perp \rangle = \langle e(\underline{E} + \underline{v} \wedge \underline{B})_\perp \rangle = -j \frac{e\beta I}{2\pi R} \underline{\underline{Z}}_\perp \cdot \Delta \underline{z}_\perp$$ 2D

Suppose the TENSOR impedance $\underline{\underline{Z}}_\perp$ is diagonal in a tilted frame

$$\underline{\underline{Z}}'_\perp = \begin{pmatrix} Z^{(1)} & 0 \\ 0 & Z^{(2)} \end{pmatrix}$$ in a frame rotated by angle α around z

In this frame the force $\underline{F}_\perp$ and the transversed displacement $\Delta \underline{z}_\perp = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$ become

$$\langle \underline{F}'_\perp \rangle = \underline{\underline{R}} \cdot \langle \underline{F}_\perp \rangle = -j \frac{e\beta I}{2\pi R} \underline{\underline{Z}}'_\perp \cdot \underline{\underline{R}} \cdot \Delta \underline{z}_\perp$$

Therefore

$$\begin{cases} \langle F_\perp \rangle = -\dfrac{j e\beta I}{2\pi R} \left(\underline{\underline{R}}^{-1} \cdot \underline{\underline{Z}}'_\perp \cdot \underline{\underline{R}} \right) \cdot \Delta \underline{z}_\perp \\ \underline{\underline{R}} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \text{ : rotation matrix} \end{cases}$$

$$\underline{\underline{Z}}_\perp = \underline{\underline{R}}^{-1} \cdot \underline{\underline{Z}}'_\perp \cdot \underline{\underline{R}} = \begin{pmatrix} \cos^2\alpha\, Z^{(1)} + \sin^2\alpha\, Z^{(2)} & \sin\alpha \cos\alpha \left(Z^{(1)} - Z^{(2)} \right) \\ \sin\alpha \cos\alpha \left(Z^{(1)} - Z^{(2)} \right) & \cos^2\alpha\, Z^{(2)} + \sin^2\alpha\, Z^{(1)} \end{pmatrix}$$

$$\boxed{\text{For } \alpha = \frac{\pi}{4} \Rightarrow \underline{\underline{Z}}_\perp = \frac{1}{2} \begin{pmatrix} Z^{(1)} + Z^{(2)} & Z^{(1)} - Z^{(2)} \\ Z^{(1)} - Z^{(2)} & Z^{(1)} + Z^{(2)} \end{pmatrix}}$$

Tune shift tensor:

$$\begin{cases} \underline{\underline{\Delta Q}} = j \dfrac{N r_p}{2\pi\gamma} \dfrac{1}{Z_0 R} \underline{\underline{\beta}}_\perp^{1/2} \, \underline{\underline{Z}}_\perp \, \underline{\underline{\beta}}_\perp^{1/2} \\ \underline{\underline{\beta}}_\perp = \begin{pmatrix} \beta_x & 0 \\ 0 & \beta_y \end{pmatrix} , \end{cases}$$

ΔQ_{xy} : coupling coefficient (compensated by incoherent tune shift)

$$\begin{cases} \text{For } \alpha = \dfrac{\pi}{4} \\ \underline{\underline{\Delta Q}} = j \dfrac{N r_p}{2\pi\gamma} \dfrac{1}{Z_0 R} \begin{pmatrix} \beta_x \dfrac{Z^{(1)}+Z^{(2)}}{2} & \sqrt{\beta_x\beta_y} \dfrac{Z^{(1)}-Z^{(2)}}{2} \\ \sqrt{\beta_x\beta_y} \dfrac{Z^{(1)}-Z^{(2)}}{2} & \beta_y \dfrac{Z^{(1)}+Z^{(2)}}{2} \end{pmatrix} \end{cases}$$

For a skew collimator tilted by $\frac{\pi}{4}$

$$Z^{(2)} = \frac{1}{2} Z^{(1)}$$

Yokoya coefficient for non-collimation plane

Therefore

$$\boxed{\begin{cases} \Delta Q_x = j \dfrac{N r_p}{2\pi\gamma} \dfrac{\beta_x}{Z_0 R} \dfrac{3}{4} Z^{(1)} \\ \Delta Q_y = j \dfrac{N r_p}{2\pi\gamma} \dfrac{\beta_y}{Z_0 R} \dfrac{3}{4} Z^{(1)} \end{cases}}$$

$$\Delta Q_{xy} = j \frac{N r_p}{2\pi\gamma} \frac{\sqrt{\beta_x\beta_y}}{Z_0 R} \frac{1}{4} Z^{(1)}$$