Tune Spread and Tune shift induced by an electron-cloud

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Analytical results (1)

- Equation of motion of an electron in the bunch potential (Hp. linear force)

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- Evolution of the electron density during the bunch passage (→via Liouville Theorem)

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- Tune shift experienced by the protons

Analytical results (2)

- Different longitudinal bunch distribution:
  - Uniform profile
  - Gaussian distribution
  - ‘Sacherer’ distribution → ~ $(1-z^2)^2$

- Initially Gaussian electron distribution → (σ_0 , σ'_0)

Equation of motion of an electron in the bunch potential (1)

- Bunch distribution:

$$\rho_b(r,z) = \frac{e\lambda_b(z)}{2\pi\sigma_r^2}\, e^{-\frac{r^2}{2\sigma_r^2}} ,$$

- Field (⟶ from Gauss theorem)

$$E(r,z) = \frac{e\lambda_b(z)}{2\pi\varepsilon_0 r}\left[1 - e^{-\frac{r^2}{2\sigma_r^2}}\right]$$

- Equation of motion:

$$m_e \frac{d^2r}{dt^2} = -eE(r,z) + \frac{l^2}{m_e r^3} = -\frac{e^2\lambda_b(z)}{2\pi\varepsilon_0 r}\left[1 - e^{-\frac{r^2}{2\sigma_r^2}}\right] + \frac{l^2}{m_e r^3} ,$$

- Derivative with respect to **z = c t**:

$$r'' \equiv \frac{d^2r}{dz^2} = \frac{1}{c^2}\frac{d^2r}{dt^2}$$

Equation of motion of an electron in the bunch potential (2)

- x-component of the eq. of motion:

$$x'' = -\frac{2r_e\lambda_b(z)x}{r^2}\left[1 - e^{-\frac{r^2}{2\sigma_r^2}}\right]$$

- Approximation of linear force (for $\mathbf{r \ll \sigma_r}$):

$$x'' + \omega_e^2(z)x = 0 \qquad \omega_e^2(z) = \frac{\lambda_b(z)r_e}{\sigma_r^2}$$

- For a longitudinally uniform bunch ($\lambda_e\mathbf{(z)}=\lambda_e$):

➡ harmonic oscillator:

$$\begin{aligned} x &= x_0\cos(\omega_e z) + x_0'\frac{1}{\omega_e}\sin(\omega_e z) \\ x' &= -x_0\omega_e\sin(\omega_e z) + x_0'\cos(\omega_e z)\ . \end{aligned}$$

Electron density distribution

- In the **linear force approximation** the 2 planes are uncoupled → factorization of the density in the phase space

$$\rho(x, x', y, y', z) = \rho_x(x, x', z)\rho_y(y, y', z) .$$

- Liouville Theorem + Hp. initially Gaussian Distribution

$$\rho_x(x, x', z) = \rho_x(x_0, x_0', 0) = \frac{\sqrt{\lambda_e}}{2\pi\sigma_0\sigma_0'}\, e^{-\frac{x_0^2}{2\sigma_0^2}}\, e^{-\frac{x_0'^2}{2\sigma_0'^2}} .$$

Where (x0, x0') are obtained by inverting the solution of the **eq. of motion → $(x_0, x_0') = f\,(x,x')$**

- The density is obtained by integrating:

$$n_x(x, z) = \int_{-\infty}^{+\infty} dx' \rho_x(x, x', z)$$

Tune shift

- The field produced by the electrons is (→Gauss theorem):

$$E(r,z) = \frac{-|e|\lambda(r,z)}{2\pi\epsilon_0 r}$$

$$\lambda(r,z) = 2\pi \int_0^r n_e(r',z) r' dr'$$

- The tune shift experienced by the protons (as a function of **r** and **z**) is:

$$\Delta Q_x = \frac{1}{4\pi} \oint_C ds\, \beta(s) \Delta k_x$$

$$\Delta k_x = -\frac{e}{\gamma m_p c^2} \frac{\partial E_x}{\partial x}$$

Longitudinal Uniform Profile

- eq. of motion → harmonic oscillator:

$$x'' + \omega_e^2 x = 0$$
$$\omega_e^2 = \frac{\overline{\lambda}_b r_e}{\sigma_r^2}.$$

- (x0, x0') = f (x,x'):

$$x_0 = x\cos(\omega_e z) - x'\frac{1}{\omega_e}\sin(\omega_e z) = Cx - \frac{S}{\omega_e}x'$$
$$x_0' = x\omega_e\sin(\omega_e z) + x'\cos(\omega_e z) = \omega_e S x + C x'$$

- electron density:

$$n(x,y,z) = n_x(x,z)n_y(y,z)$$
$$= \frac{\lambda_e \omega_e^2}{2\pi}\frac{e^{-\frac{\omega_e^2 r^2}{2\left(\sigma_0'^2 S^2 + \omega_e^2\sigma_0^2 C^2\right)}}}{\sigma_0'^2 S^2 + \omega_e^2\sigma_0^2 C^2}$$

- tune shift:

$$\Delta Q_x(r,z) \approx \frac{\bar{\beta}\bar{C}\lambda_e r_p}{4\pi\gamma C^2\sigma_0^2}\frac{1}{1+\frac{S^2\sigma_0'^2}{C^2\omega_e^2\sigma_0^2}}\left(1 - \frac{\omega_e^2 r^2}{2(C^2\omega_e^2\sigma_0^2 + S^2\sigma_0'^2)}\right)$$

- Hp. σ_0'«$\boldsymbol{\sigma_0\omega_e}$; for **r = 0 ,z=0**:

$$\Delta Q_x(r,z) \approx \frac{\bar{\beta}\bar{C}\lambda_e r_p}{4\pi\gamma\sigma_0^2}$$

General longitudinal distribution

- eq. of motion: $x'' + \omega_e^2(z)x = 0$ $\qquad \omega_e^2(z) = \frac{\lambda_b(z) r_e}{\sigma_r^2}$
- solution of eq. of motion in the form:

$$x(z) = A(z)e^{iS(z)}$$

- **WKB approximation**:

$$\left|\frac{3}{2}\frac{\omega_e'}{\omega_e}\right|, \quad \left|\frac{\omega_e''}{\omega_e'}\right| \ll 2\omega_e \quad \Rightarrow \quad A(z) \approx \frac{1}{\sqrt{S'(z)}}, \quad S'(z) \approx \omega_e(z)$$

- The general solution can also be written as:

$$x(z) = \frac{A}{\sqrt{\omega_e(z)}}\cos\left[S(z)\right] + \frac{B}{\sqrt{\omega_e(z)}}\sin\left[S(z)\right]$$

General longitudinal distribution

- eq. of motion:

$$x'' + \omega_e^2(z) x = 0 \qquad \omega_e^2(z) = \frac{\lambda_b(z) r_e}{\sigma_r^2}$$

- (x0, x0') = f (x,x'):

$$\begin{aligned} x_0 &= a(z)x + b(z)x' \\ x_0' &= c(z)x + d(z)x' \end{aligned}$$

- electron density:

$$n(r,z) = \frac{\lambda_e}{2\pi[d(z)^2\sigma_0^2 + b(z)^2\sigma_0'^2]} e^{-\frac{r^2}{2[d(z)^2\sigma_0^2 + b(z)^2\sigma_0'^2]}}$$

- tune shift:

$$\Delta Q_x(r,z) \approx \frac{\bar{\beta}\bar{C}\lambda_e r_p}{4\pi\gamma\,[d(z)^2\sigma_0^2 + b(z)^2\sigma_0'^2]}\left(1 - \frac{3}{4}\frac{r^2}{[d(z)^2\sigma_0^2 + b(z)^2\sigma_0'^2]}\right) + O\left(\frac{r}{\sigma_0}\right)^4$$

‘Sacherer’ distribution

- bunch longitudinal profile:

$$\lambda_b(z) = \frac{15N_b}{8l_0}\left[1-\left(\frac{2z}{l_0}\right)^2\right]^2 \quad ; \quad z \in \left(-\frac{l_0}{2}, +\frac{l_0}{2}\right)$$

- electron density:

$$n(r,z) = \frac{\lambda_e\left(1-\frac{4z^2}{l_0^2}\right)}{2\pi\left\{\cos^2[\theta(z)]\,\sigma_0^2 + \frac{8\sigma_r^2 l_0}{15 r_e N_b}\sin^2[\theta(z)]\,\sigma_0'^2\right\}} e^{-\frac{r^2\left(1-4z^2/l_0^2\right)}{2\left\{\cos^2[\theta(z)]\sigma_0^2+\frac{8\sigma_r^2 l_0}{15 r_e N_b}\sin^2[\theta(z)]\sigma_0'^2\right\}}}$$

$$\theta(z) = \sqrt{\frac{15 r_e N_b}{8\sigma_r^2 l_0}}\left(z-\frac{4z^3}{3l_0^2}\right)$$

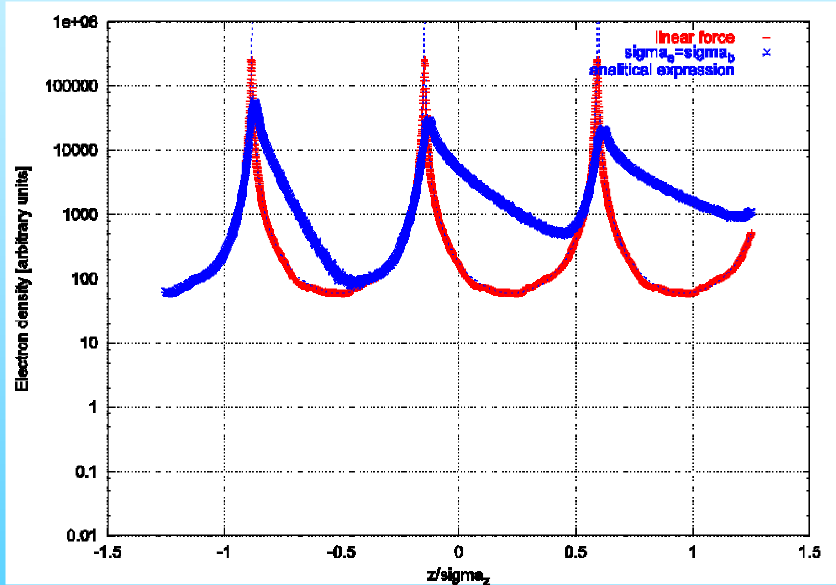
- tune shift:

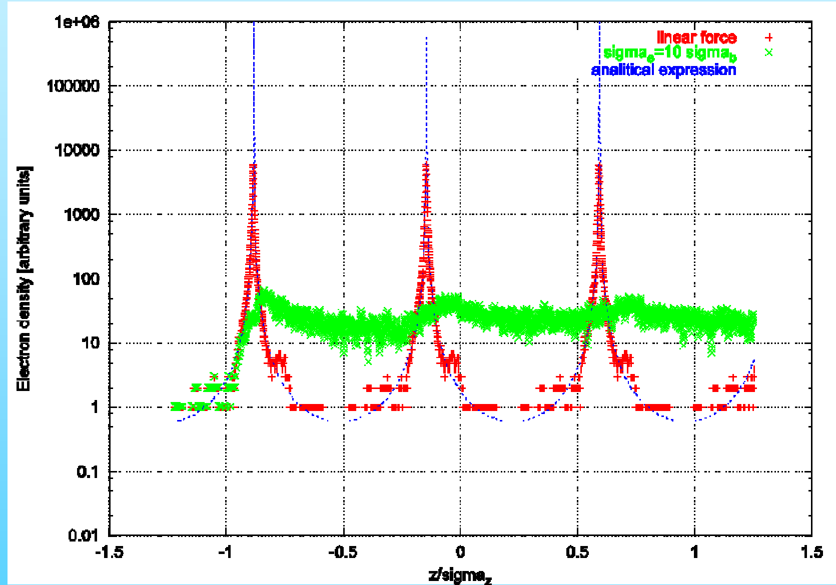
$$\Delta Q_x(r,z) \approx \frac{\bar{\beta}\bar{C}\lambda_e r_p\left(1-\frac{4z^2}{l_0^2}\right)}{4\pi\gamma\left[\cos^2[\theta(z)]\,\sigma_0^2 - \frac{8\sigma_r^2 l_0}{15 r_e N_b}\sin^2[\theta(z)]\,\sigma_0'^2\right]}\left(1-\frac{3}{4}\frac{\left(1-\frac{4z^2}{l_0^2}\right)r^2}{\left(\cos^2[\theta(z)]\,\sigma_0^2 - \frac{8\sigma_r^2 l_0}{15 r_e N_b}\sin^2[\theta(z)]\,\sigma_0'^2\right)}\right)$$

- Hp. σ_0’«$\boldsymbol{\sigma_0 \omega_e}$; for **r = 0 , z=0**:

$$\Delta Q_x(r,z) \approx \frac{\bar{\beta}\bar{C}\lambda_e r_p}{4\pi\gamma\sigma_0^2}$$

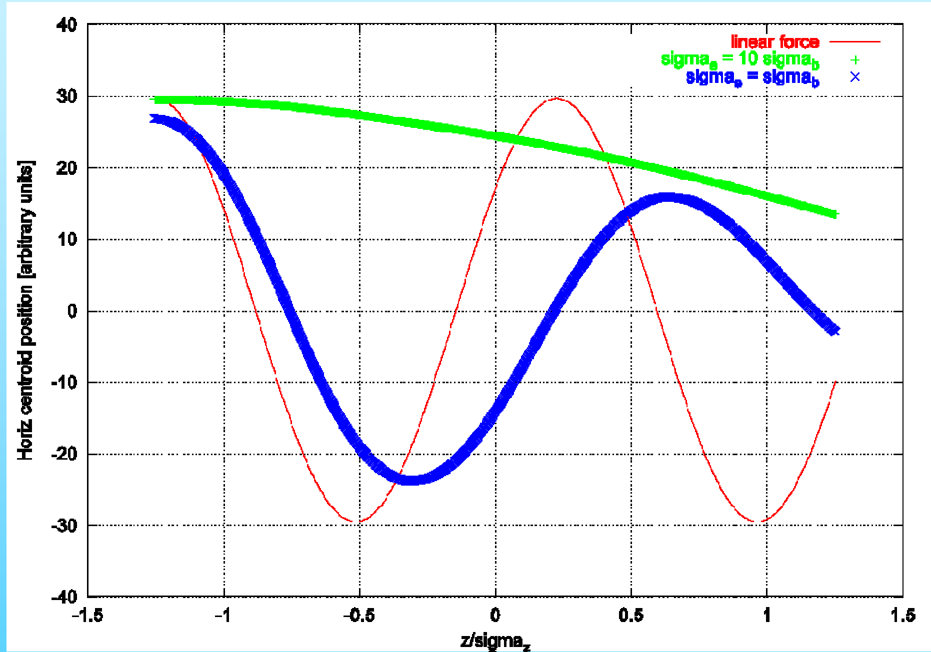
Simulations: Gaussian force + longitudinally uniform bunch





Electron density vs. time at the centre r=0 for an electron size = sigma_b (Left) and 10*sigma_b (Right)

Simulations: Gaussian force + longitudinally uniform bunch



Horizontal centroid position Vs. time for an electron size=10 sigma_b (green) and sigma_b (blue), for an initial offset of 1/sigma_b