SYNCHRONOUS PHASE AND INCOHERENT FREQUENCY SHIFTS

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- **Suspicions from Elena of some inconsistencies between different papers**

Consider a Gaussian bunch

$$I(t) = \frac{q}{\sqrt{2\pi}\,\sigma_t}\, e^{-\frac{t^2}{2\sigma_t^2}}$$

$$q = N_b\, e$$

$$I_b = q\, f_0$$

$$\int_{t=-\infty}^{t=+\infty} I(t)\,dt = q$$

$$\tilde{I}(\omega) = \int_{t=-\infty}^{t=+\infty} I(t)\, e^{-j\omega t}\, dt \quad \Rightarrow \quad \tilde{I}(\omega) = q\, e^{-\frac{(\omega\sigma_t)^2}{2}}$$

$$\tilde{\lambda}(\omega) = e^{-\frac{(\omega\sigma_t)^2}{2}}$$

$$h_m(\omega, \sigma_t) = \left| \tilde{\lambda}_m(\omega) \right|^2 = (\omega\sigma_t)^{2m}\, e^{-(\omega\sigma_t)^2}$$

$$= \text{spectral power density}$$

◆ From Laclare (CAS CERN 87-03)

- **Synchronous phase shift**

$$\Delta\phi_s = \frac{2 I_b}{\hat{V}_{RF} \cos\phi_{s0}} \sum_{p=0}^{p=+\infty} \mathrm{Re}\left[Z_l \left(p\,\omega_0 \right) \right] e^{-\frac{1}{2}\left(p\,\omega_0\,\sigma_t \right)^2}$$

- **Longitudinal loss factor (defined elsewhere)**

$$k_l\left(\sigma_t\right) = \frac{\omega_0}{\pi} \sum_{p=0}^{p=+\infty} \mathrm{Re}\left[Z_l \left(p\,\omega_0 \right) \right] e^{-\left(p\,\omega_0\,\sigma_t \right)^2}$$

- **Relation between the 2**

$$\Delta\phi_s = \frac{N_b\, e}{\hat{V}_{RF} \cos\phi_{s0}} k_l\left(\frac{\sigma_t}{\sqrt{2}} \right)$$

◆ **From Zotter (Handbook Chao-Tigner, 2nd printing, p. 118)**

- **Relation between the longitudinal effective impedance and the impedance Z_{hom} from Zotter**

$$Z_{\text{hom}} = \sum_{p=-\infty}^{p=-\infty} \text{Re}\left[Z_l\left(p\,\omega_0\right)\right] \tilde{\lambda}\left(p\,\omega_0\right) = \frac{\sqrt{2\,\pi}}{\omega_0\,\sigma_t} \text{Re}\left[\left(\frac{Z_l}{n} \right)^{eff}_{m=0} \right]$$

?

- **Relation between the longitudinal loss factor and the impedance Z_{hom} from us (?)**

$$Z_{\text{hom}} = \sum_{p=-\infty}^{p=-\infty} \text{Re}\left[Z_l\left(p\,\omega_0\right)\right] \tilde{\lambda}\left(p\,\omega_0\right) = T_0\, k_l \left(\frac{\sigma_t}{\sqrt{2}} \right)$$

$\Rightarrow$ $\Delta\,\phi_s = \text{same as Laclare}$ **(suspected typo errors in the Handbook if we did not make mistakes…)**

- **Relation between the longitudinal effective impedance and the impedance Z_{pot}**

$$Z_{\text{pot}} = \sum_{p=-\infty}^{p=+\infty} \text{Im}\left[Z_l \left(p\,\omega_0 \right) \right] p\, \tilde{\lambda} \left(p\,\omega_0 \right)$$

$$= \frac{\sqrt{2\pi}}{\left(\omega_0\, \sigma_t \right)^3} \text{Im}\left[\left(\frac{Z_l}{n} \right)_{m=1}^{eff,\,\omega_s = 0,\,\sigma_t' = \frac{\sigma_t}{\sqrt{2}}} \right]$$

$$\left[\frac{Z_l}{n} \right]_m^{eff} = \frac{\displaystyle\sum_{p=-\infty}^{p=+\infty} \frac{Z_l\left(\omega_p^l \right)}{p} \left(\frac{\omega_p^l}{\omega_0} \right)^{2m} e^{-\left(\omega_p^l\, \sigma_t \right)^2}}{\displaystyle\sum_{p=-\infty}^{p=+\infty} \left(\frac{\omega_p^l}{\omega_0} \right)^{2m} e^{-\left(\omega_p^l\, \sigma_t \right)^2}}$$

$$\omega_p^l = p\,\omega_0 + m\,\omega_s$$

- From Chao's book ("Physics of Collective…"), p. 291, Eq. (6.59)

$\Rightarrow$ He has the same kind of behaviour as Laclare and Zotter. It is the Fourier transform of the line density which enters into the equation

- **From Hofmann (CAS CERN 95-06)**

$$\Delta \phi_s = \frac{N_b \, e}{\hat{V}_{RF}} \, k_l \left(\sigma_t \right)$$

And not $\sigma_t / \sqrt{2}$ as Laclare, Zotter and Chao…

⇒ **For him it is not the Fourier transform of the line density which enters into the equation but the spectral power density !**

⇒ **And same thing for the incoherent frequency shift !**

(suspected typo errors in the paper if we did not make mistakes…)

- **In his computation, A. Hofmann starts with a stationary bunch having no synchrotron oscillations**

$$I_k(t) = \sum_{k=-\infty}^{k=+\infty} I(t - k T_0)$$

- **Then he computes the induced voltage by the stationary bunch in the presence of a general impedance**

$$\widetilde{V}_k(\omega) = \widetilde{I}_k(\omega) Z_l(\omega)$$

$$\Rightarrow \quad V_k(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widetilde{V}_k(\omega) e^{j\omega t} d\omega = I_b \sum_{p=-\infty}^{p=+\infty} Z_l(p\omega_0) \widetilde{\lambda}(p\omega_0) e^{j p \omega_0 t}$$

(I think the time *t* in the exponential is replaced by τ if the synchrotron oscillations are taken into account (see Laclare). Then one can expand the exponential in series if one considers small-amplitude synchrotron oscillations… and one would obtain the same result as Laclare…)

- **A. Hofmann averages his voltage over time**

$$< V > = \frac{1}{I_b \, T_0} \int_{t=-T_0/2}^{t=+T_0/2} I_k(t) \, V_k(t) \, dt$$?

$$\Rightarrow \quad < V > = I_b \sum_{p=-\infty}^{p=+\infty} \mathrm{Re}\left[Z_l \left(p \, \omega_0 \right) \right] \left| \tilde{\lambda} \left(p \, \omega_0 \right) \right|^2$$

and $$\sin \phi_s = \frac{< V >}{\hat{V}_{RF}}$$

CONCLUSION

- What do you think ?

  - For me, the derivation by Laclare is consistent, and one cannot relate the synchronous phase shift with the loss factor (for the same bunch length) or effective impedance

  - In the derivation from A. Hofmann, he does not take into account the synchrotron oscillations and makes and averaging, which I don't understand for the moment (but which is certainly right…). He can then relate the synchronous phase shift to the loss factor and the incoherent frequency shift to the effective impedance

- This is not purely academic as we are trying to understand the loss of Landau damping in the SPS and beam loss due to intensity effects…