Transverse Impedance of Resistive Metal Walls of Finite Thickness

Bruno Zotter

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Source Models

- Source models describe transversely oscillating beam by simpler geometry which can be treated analytically
- Nearly 50 years ago, in seminal paper on transverse resistive wall effect (Laslett, Neil, and Sessler 1965), transverse oscillations of continuous beam with uniform transverse distribution were replaced by surface charges and currents. They derived "standard" expression for transverse impedance of cylindrical resistive vacuum chamber with radius b, length L, conductivity σ, wall thickness large compared to skin depth $\delta = \sqrt{2/\omega\mu\sigma}$:

$$Z_{\perp}(\omega) = (1+j) Z_0 \frac{\mu_r L}{2\,\pi\, b^3} \delta(\omega) \qquad (1)$$

where $Z_0 = 120\pi\Omega$, $\mu = \mu_r \mu_0$ is the magnetic permeability. Skin depth factor makes impedance proportional to $1/\sqrt{\omega}$.

- when skin depth less than wall thickness, impedance becomes $\propto 1/\omega$ - i.e. increases even faster at low frequencies (Sacherer Erice 1976) "provided there are no alternate paths for the induced current"
- For evaluation of transverse impedance of metalized ceramic chambers in the ISR I used Sessler's model (CERN Report 69-15, also described in book with S.Kheifets, chapter6.5). Unfortunately, matrix technique I used to evaluate expressions not very transparent, and model not useful for non-uniform or bunched beams.

  It was basis for computer program LAWAT for LAminated WAll Transverse impedance for arbitrary number of layers. To avoid self-made subroutines for Bessel functions of complex arguments it was rewritten with MATHEMATICA in the 1980-ies and applied to SPS and VLHC by O.Meincke and E.Keil, who organized"'packages" to expedite calculations.

- Approximate expression for impedance of thin metal wall , showed frequency dependence of real part with decreasing frequency - after passing through a maximum- changes from $\propto 1/\omega$ to $\propto \omega$ when frequency approaches zero, i.e. induced currents always flow in structures outside wall proper (sometimes called “ redistribution effect” or “inductive bypass” if only vacuum outside).

- Another source model introduced by A. Chao (Physics of collective effects) and R. Gluckstern (US Accelerator School, CERN Report 2000-11): infinitesimal thin ring with $\cos(m\theta)$ modulation, $m = 1$ corresponding to dipole term in field expansion of offset beam). Better adapted to non-uniform or bunched beams. Started rewrite of LAWAT as LAWAT2000. in order to compare both models.

- Similar model used by Burov-Lebedev: oscillating electric and magnetic dipoles. From beginning authors assume electric fields do NOT penetrate chamber wall ("electro-static screening"), furthermore exclude TE modes when assuming purely longitudinal vector potential. In cylindrical coordinates,

$$B_z = \frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \qquad (2)$$

thus B_z vanishes when $A_r = A_\theta = 0$, hence no TE mode.

- Although TE modes **not** directly excited by oscillating beam, needed to fulfill boundary conditions at walls of finite resistivity. Coupling of TM and TE modes in resistive wave guide already shown in text book by Stratton[**?**].
- These rather restrictive assumptions simplify analysis considerably as matching at each boundary reduced to only 2 rather than 4 tangential field components. Claimed valid when b small compared to betatron wave length. While this is almost always true, need verify.for various geometries; Recently found o.k. for LHC collimators, but not for [**?**] MKE kickers.

- L Vos used circuit theory to calculate longitudinal impedance. of parallel resistance and inductance of chamber in free space $\mu_0/4\pi$ (value not justified), then transforms it to transverse impedance with approximate relation $Z_\perp = 2c/b^2 Z_{||}/\omega$.
- In second report he extends circuit theory approach to multi-layer walls, but still uses he approximate relation between modes with different azimuthal mode numbers correct for direct space direct charge impedance where $Z_{||}(s.c.) \propto log(a/b)$, while $Z_\perp(s.c.) \propto (1/a^2 - 1/b^2)$. Analysis gives in general total impedance, containing both space charge and wall impedance terms, and thus validity of transformation at least limited to very high γ where direct space charge contribution become negligible.

- **LHC collimators**, made of highly resistive graphite, can be moved very close to beam (smallest distance b down to 1.2 mm for some of them). Due to factor $1/b^3$ - and also due to low conductivity of graphites - standard formula would give very high transverse impedance, in particular at frequency of slowest betatron wave $(n - Q)f_0$, about 8 KHz in LHC.
- This would cause limitations of beam current and hence collider performance, thus a number of papers[**?**, **?**, **?**, **?**, **?**] were recently published on this subject, but results do not always agree in region of interest

Wave or Helmholtz Equation

- Electro-magnetic fields excited by the model source can be found by solving wave equation (Helmholtz equation in frequency domain) either for vector or Hertz potentials, or directly for longitudinal field components (the transverse components are more difficult to obtain directly as the transverse components of the (vector) wave equation are coupled). From Maxwell equations, transverse field components can be obtained rather simply from the longitudinal ones and their derivatives (see below).
- Assuming frequency dependence $\propto \exp(j\omega t)$, and combining conduction $\vec{J}_c = \sigma_c \vec{E}$ with displacement current density $\vec{J}_d = \omega\varepsilon\vec{E}$ as $J_{tot} = j\omega\varepsilon_c\vec{E}$ with *"complex permittivity"* $\varepsilon_c = \varepsilon'\varepsilon_0$ where $\varepsilon' = \varepsilon_r - j\sigma_c/(\omega\varepsilon_0)$. For metals , ε_r usually taken as unity (not well known), imaginary part anyhow dominant..

- The (vector) Helmholtz equations can be written

$$\begin{aligned} \left[\Delta - \varepsilon_c \mu \omega^2\right] \vec{E} &= \frac{1}{\varepsilon} \operatorname{grad} \rho - j\omega\mu\rho\,\vec{v}\,, \\ \left[\Delta - \varepsilon_c \mu \omega^2\right] \vec{H} &= -\rho \operatorname{curl} \vec{v}\,. \end{aligned} \tag{3}$$

  where $\Delta = \nabla^2$ is *Laplacian operator*. Same equations in scalar form for axial components E_z and H_z.

- In a source-free region, charge density $\rho = 0$, thus RHSs of both equations zero, sufficient to solve homogeneous Helmholtz equation

Product solutions of Helmholtz Equation

- Solution found readily as products of functions in the 3 variables r, θ, and z. This yields harmonic oscillator equations for θ and z dependent functions, with solutions $\exp(\pm jm\theta)$ and $\exp(\pm jkz)$. In order for the solutions to be single valued, m must be an integer, *azimuthal mode number*: $m = 1$ for dipole fields..
- The *axial mode number* $n = 2\pi R/\lambda$ is number of betatron wavelengths around circumference, therefor *wave number* becomes $k = 2\pi/\lambda = n/R$.
- Axial and time dependence combined are $\exp[j(\omega t - kz)]$ Betatron frequency (in beam frame) $\omega = Q\omega_0$, where $Q = [Q] + q$, $[Q]$ integer, q non-integer parts of betatron tune Frequency at fixed location $(z - \beta ct = z - R\omega_0 t)$ becomes $(Q + n)\omega_0$.

- Since n can be negative, often written $Q \pm n$ with $(n > 0)$. Plus sign refers to *fast waves*, propagating in direction of beam, minus sign to *slow waves*, propagating in opposite direction. Only these can become unstable[a]. Wave velocity

$$v_w = \beta_w c = \omega/k = \beta c \left[1 \pm \frac{Q}{n}\right], \tag{4}$$

may differ considerably from beam velocity, (much) smaller for slow waves. in particular when n near $[Q]$. For $q < 1/2$, lowest wave velocity is $\beta_b cq/[Q]$ (replace q by $(1-q)$ when $q > 1/2$). For LHC: $q \approx 0.3$, $[Q] \approx 70$, and thus $\beta_{min} \approx 0.04$!

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[a]slow waves have "negative energy", i.e. amplitudes grow exponentially when coupled to a (positive energy) wave on surrounding structure, fast waves only "beating, i.e. periodic in- and decrease of amplitudes.

Field calculations

- Radial part of homogeneous Helmholtz equation: differential equation for (modified) Bessel functions of m-th order and argument νr, where the *radial propagation constant* ν is given by separation constant

$$\nu^2 = k^2 - \omega^2 \mu \varepsilon_c = k^2 (1 - \beta^2 \varepsilon' \mu') \,. \tag{5}$$

  The sign of the square root for ν has to be chosen such that the solutions decay for $r \to \infty$.

- Longitudinal field components in n-th region superpositions of modified Bessel functions of first and second kind

$$\begin{aligned} E_z^{(n)}(r) &= [E_n K_1(\nu_n r) + F_n I_1(\nu_n r)] \cos\theta \\ H_z^{(n)}(r) &= [G_n K_1(\nu_n r) + H_n I_1(\nu_n r)] \sin\theta \,, \end{aligned} \tag{6}$$

  where E_n, F_n, G_n, H_n coefficients determined by matching.

Avoid excessive values for large arguments of modified Bessel functions, better normalize to values at boundary; also introduce $\alpha_n = -F_n/E_n$ and $\eta_n = -H_n/G_n$ to get

$$\begin{aligned} E_z^{(n)}(r) &= E_n \left[\frac{K_1(\nu_n r)}{K_1(\nu_n b)} \alpha_n \frac{I_1(\nu_n r)}{I_1(\nu_n b)} \right] \cos\theta \,, \\ H_z^{(n)}(r) &= G_n \left[\frac{K_1(\nu_n r)}{K_1(\nu_n b)} - \eta_n \frac{I_1(\nu_n r)}{I_1(\nu_n b)} \right] \sin\theta \,. \end{aligned} \tag{7}$$

Transverse Field Components

- Using complex permittivity $\varepsilon_c = \varepsilon' \varepsilon_0 = \varepsilon_r + \sigma/(j\omega)$ and complex permeability $\mu_c = \mu' \mu_0 = \mu_r(1 + j \tan \Theta_M$, in cylindrical coordinates, r and θ components of Maxwell curl equations $curl \vec{H} = j\omega\varepsilon_c \vec{E}$, $curl \vec{E} = -j\omega\mu_c \vec{H}$ are

$$
\begin{aligned}
\frac{1}{r}\frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z} &= j\omega\varepsilon_c E_r \,, \\
\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} &= j\omega\varepsilon_c E_\theta \,, \\
\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z} &= -j\omega\mu_c H_r \,, \\
\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} &= -j\omega\mu_c H_\theta \,. \qquad (8)
\end{aligned}
$$

- Abbreviating $\vec{G} = Z_0 \vec{H}$ (same dimension as electric field strength). Reorder to get 2 sets of linear equations with 2 unknowns each:

$$
\begin{aligned}
j\omega\varepsilon' E_r - jkG_\theta &= \frac{G_z}{r}\,, \\
-\,jkE_r + j\omega G_\theta &= \frac{dE_z}{dr} \\
j\omega\varepsilon' E_\theta - jkG_r &= -\frac{dG_z}{dr}\,, \\
-\,jkE_\theta + j\omega G_r &= \frac{E_z}{r}\,,
\end{aligned} \tag{9}
$$

Both sets same determinant $k^2 - \omega^2\varepsilon'\mu'/c^2 = \nu^2$,

- Solutions for azimuthal field components:

$$
\begin{aligned}
E_\theta &= -\frac{jk}{\nu^2}\left[\frac{E_z}{r} + \beta\,\mu'\frac{dG_z}{dr}\right]\,, \\
G_\theta &= \frac{jk}{\nu^2}\left[\frac{G_z}{r} + \beta\,\varepsilon'\frac{dE_z}{dr}\right]\,.
\end{aligned} \tag{10}
$$

Transverse Impedance

- Usual definition of (horizontal) transverse impedance

$$Z_\perp(\omega) = \frac{j}{P}\int_{-\infty}^{\infty} dz\,[E_x - v_z B_y]\,\mathrm{e}^{jkz}\ , \tag{11}$$

where P is the dipole moment of the charge distribution in the beam. All field components proportional to P, drops out in calculation.

- Rewritten in terms of transverse derivative of longitudinal electric field strength[**?**]

$$Z_\perp(\omega) = -\frac{1}{kP}\int_{-\infty}^{\infty} dz\frac{\partial E_z}{\partial x}\mathrm{e}^{jkz} = -\frac{1}{P^2}\int dV E_z J_z^*\ . \tag{12}$$

Advantage: magnetic field strength not needed explicitly.

Complex conjugate of source current density of dipole modulated ring beam, radius a

$$J_z^* = \frac{P}{\pi a^2}\delta(r-a)\cos\theta \mathrm{e}^{jkz}\,.. \tag{13}$$

- Longitudinal electric field strength
  $E_z = jC[K_1(s) - \alpha_{TM}I_1(s)]cos\theta.$
  With $s = ka/\gamma \ll 1$ $E_z = jC[(1/s) - s\alpha_{TM}/2]\cos\theta$ where coefficient

$$C = \frac{\omega P}{\pi\varepsilon_0 v^2\gamma^2}\, I_1(s)\mathrm{e}^{-jkz}\,. \tag{14}$$

- Integral over z simply $L(=2\pi R)$, over θ simply π. over r (delta function): electric field strength evaluated at beam radius $r = a$:

$$Z_\perp(\omega) = -\frac{jLZ_0}{2\pi\beta\gamma^2}\left[\frac{1}{a^2} - \alpha_{TM}\frac{k^2}{2\gamma^2}\right]. \tag{15}$$

- Subtract and add $1/b^2$ in bracket, recognize *direct pace charge term* for beam with radius a in perfectly conducting wall at

$r = b$:

$$Z_\perp^{SC} = -\frac{jLZ_0}{2\pi\beta\gamma^2}\left[\frac{1}{a^2} - \frac{1}{b^2}\right], \tag{16}$$

(disappears for $\gamma \to \infty$).

- Contribution due to wall resistivity

$$Z_\perp^{RW} = -\frac{jLZ_0}{2\pi b^2}\left[\alpha_{TM}\frac{k^2b^2}{2\gamma^2}\right]. \tag{17}$$

- Introduce modified coefficient
  $\alpha' = \alpha_{TM} I_1(x)/K_1(x) \approx \alpha_{TM} k^2b^2/(2\gamma^2)$ to get

$$Z_\perp^{RW} = -\frac{jLZ_0}{2\pi b^2}\frac{\alpha'}{\beta\gamma^2}. \tag{18}$$

Remains finite for $\gamma \to \infty$ since α' proportional γ^2.

Conducting wall with finite thickness

- Region I ($a < r < b$) (vacuum outside beam): $\varepsilon' = \mu' = 1$, $\nu = k/\gamma \ll 1/b$ for large γ. Axial field components

$$
\begin{aligned}
E_z^{(1)}(r) &= E_1 \frac{K_1(\nu r)}{K_1(\nu b)} + F_1 \frac{I_1(\nu r)}{I_1(\nu b)}\,, \\
G_z^{(1)}(r) &= G_1 \frac{K_1(\nu r)}{K_1(\nu b)} + H_1 \frac{I_1(\nu r)}{I_1(\nu b)}\,.
\end{aligned}
\tag{19}
$$

At wall radius $E_z^{(1)}(b) = E_1 + F_1$ and $G_z^{(1)}(b) = G_1 + H_1$.

Source fields

- The fields of an infinitesimally thin ring with cosine modulation $\propto \cos\theta$ can be written[**?**]

$$
\begin{aligned}
E_z^{(s)}(r) &= jC\cos\theta F_1(u)\,, \\
E_{\theta(s)}(r) &= \gamma C\sin\theta\left[\frac{F_1(u)}{u} + \beta\alpha_{TE}\frac{dI_1(u)}{du}\right]\,, \\
Z_0 H_z^{(s)}(r) &= jC\sin\theta\alpha_{TE} I_1(u)\,, \\
Z_0 H_\theta^{(s)}(r) &= -\gamma C\cos\theta\left[\beta\frac{dF_1(u)}{du} + \alpha_{TE}\frac{dI_1(u)}{du}\right]\,, \quad (20)
\end{aligned}
$$

where $F_1(u) = K_1(u) - \alpha_{TM} I_1(u)$ and $u = kr/\gamma$.

Comparing the expressions for the fields in region I (vacuum outside beam, $a < r < b$) with source fields shows

$G_1 = 0\,, E_1 = H_1 = jC = 1$, may be normalized to unity since all field components proportional to it. With $F_1/E_1 = \alpha$ and

$H_1/G_1 = \eta_1 \; E_z^{(1)}(b) = 1 - \alpha = \alpha'$ and $G_z^{(1)}(b) = \eta$.

- For large γ, $kb/\gamma \ll 1$, replace Bessel functions by small argument approximations $I_1(z) = z/2,\; K_1(z) = 1/z$ to get

$$
\begin{aligned}
E_\theta^{(1)}(b) &= \alpha' + \beta\eta\,, \\
G_\theta^{(1)}(b) &= \beta\alpha' + \eta - 2\beta
\end{aligned}
\qquad (21)
$$

- Region II ($b < r < d$). wall material $\varepsilon' = \varepsilon_r - j\sigma/\omega\varepsilon_0;\mu'$,
  $\nu = k\sqrt{1-\beta^2\varepsilon'\mu'} \approx (1+j)/\delta$;

  field components functions of frequency ω and wave number $k = \omega/\beta c$ (β is wave velocity);.

  axial field components

$$\begin{aligned} E_z^{(2)}(r) &= E_2\left[\frac{K_1(\nu r)}{K_1(\nu b)} - \alpha_2\frac{I_1(\nu r)}{I_1(\nu b)}\right]; \\ G_z^{(2)}(r) &= G_2\left[\frac{K_1(\nu r)}{K_1(\nu b)} - \eta_2\frac{I_1(\nu r)}{I_1(\nu b)}\right]. \end{aligned} \qquad (22)$$

- Normalization to inner boundary avoids excessive numerical values for terms with modified Bessel functions of large argument; then $E_z(b) = E_2(1-\alpha_2)$ and $G_z(b) = G_2(1-\eta_2)$.

- Azimuthal field components at $r = b$ with $p = k^2\delta^2/2, q = kb, r = \beta\mu' k\delta$ become

$$
\begin{aligned}
E_\theta^{(2)}(b) &= -\frac{p}{q}\left[E_2(1-\alpha_2) + \beta\mu' G_2(Q_2 - \eta_2 P_2)\right]; \\
G_\theta^{(2)}(b) &= \frac{p}{q}\left[G_2(1-\eta_2)\beta\varepsilon' E_2(Q_2 - \alpha_2 P_2)\right]:
\end{aligned}
\tag{23}
$$

where $P_2 = I_1'(\nu b)/I_1(\nu b)$ and $Q_2 = K_1'(\nu b)/K_1(\nu b)$.

Coefficients α_2 and η_2 depend on conditions at outer boundary at $r = d$,

Distinguish 3 cases:

- (a) outer region to perfect conductor $E_z(d) = E_\theta(d) = 0$, hence also $dG_z/dr = 0$ and $\alpha_2 = K_1(\nu d)I_1(\nu b)/K_1(\nu b)I_1(\nu d)$;
- (b) outer region to perfect magnet: $G_z(d) = G_\theta(d) = 0$; hence also $dE_z/dr = 0$ and $\alpha_2 = K_1'(\nu d)I_1(\nu b)/K_1(\nu b)I_1'(\nu d)$;
- (c) outer region vacuum to infinity: $\alpha_2 = \eta_2 = 0$.

- matching axial field components gives simple relations $E_2(1-\alpha_2) = \alpha'$ and $G_2(1-\eta_2) = \eta$.
- . Substitution into azimuthal field matching equations yields

$$\begin{aligned}\alpha'\left[\gamma^2 + jp\right] + \eta\left[\beta\gamma^2 - (1-j)\frac{r}{2}qQ_\eta\right] &= 0 \\ \alpha'\left[\beta\gamma^2 + (1+j)\frac{q}{r}Q_\alpha\right] + \eta\left[\gamma^2 + jp\right] &= 2\beta\gamma^2\,. \qquad (24)\end{aligned}$$

Two linear equations with two unknowns α' and η

- Substituting solution for α', transverse resistive wall impedance becomes

$$Z_\perp(\omega) = j\frac{LZ_0}{\pi b^2}\frac{1-(1-j)qQ_\eta\dfrac{r}{2\beta\gamma^2}}{1+2jp-\frac{p^2}{\gamma^2}-\beta\left[(1+j)\frac{q}{r}Q_\alpha+(1-j)qQ_\eta\frac{r}{2}\right]}\ , \tag{25}$$

where

$$Q_\alpha = \frac{Q_2-\alpha_2 P_2}{1-\alpha_2} \qquad Q_\eta = \frac{Q_2-\eta_2 P_2}{1-\eta_2} \tag{26}$$

For large γ this expression simplifies to

$$Z_\perp(\omega) = j\frac{LZ_0}{\pi b^2}\frac{1}{1-(1-j)qQ_\eta r/2} \tag{27}$$

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