IBS studies

- CERN experiments at low or moderate energy are said to disagree with MAD predictions (J.-Y. Hemery)
- Michel Martini & Frank Schmidt recommend implementation of the Conte-Martini formulae, which are non-ultrarelativistic generalization based on Bjorken-Mtingwa
- check of algorithm implemented in MAD
- extension to include vertical dispersion which is important for damping rings

IBS growth rates in general Bjorken-Mtingwa theory

$$\frac{1}{\tau_a} = \frac{\pi^2 \alpha^2 MN(\log)}{\gamma\tilde{\Gamma}} \left\langle \int_0^\infty \frac{d\lambda\, \lambda^{1/2}}{[\det(\mathrm{L}+\lambda \mathrm{I})]^{1/2}} \left\{ \mathrm{Tr} L^{(a)} \mathrm{Tr}\left(\frac{1}{L+\lambda I}\right) - 3\mathrm{Tr} L^{(a)}\left(\frac{1}{L+\lambda I}\right)\right\}\right\rangle$$

where

$$L = L^{(h)} + L^{(l)} + L^{(v)}$$

$$L^{(h)} = \frac{\beta_x}{\varepsilon_x}\begin{pmatrix} 1 & -\gamma\phi_x & 0 \\ -\gamma\phi_x & \gamma^2 H_x/\beta_x & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad L^{(l)} = \frac{\gamma^2}{\sigma_\delta^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad L^{(v)} = \frac{\beta_z}{\varepsilon_z}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma^2 H_z/\beta_z & -\gamma\phi_z \\ 0 & -\gamma\phi_z & 1 \end{pmatrix}$$

I introduced vertical dispersion here

$$\phi_{x,z} = D'_{x,z} - \frac{\beta'_{x,z} D_{x,z}}{2\beta_{x,z}}$$

Bjorken-Mtingwa [1] gave solution in ultrarelativistic limit;
Conte-Martini [2] kept all terms
In the limit D_z=0, I should recover CM result

[1] J.D. Bjorken, S.K. Mtingwa, “Intrabeam Scattering,” Part. Acc. Vol. 13, pp. 115-143 (1983).
[2] M. Conte, M. Martini, “Intrabeam Scattering in the CERN Antiproton Accumulator,” Part. Acc. Vol. 17, pp. 1-10 (1985).

The general form of all solutions is

$$\frac{1}{\tau_x} = \frac{\pi^2 \alpha^2 MN(\log)}{\gamma\tilde{\Gamma}} \left[\frac{\gamma^2 H_x}{\varepsilon_x} \right] \int_0^\infty \frac{d\lambda\, \lambda^{1/2} \left[a_x \lambda + b_x \right]}{\left\{ \lambda^3 + a\lambda^2 + b\lambda + c \right\}^{3/2}}$$

$$\frac{1}{\tau_l} = \frac{\pi^2 \alpha^2 MN(\log)}{\gamma\tilde{\Gamma}} \left[\frac{\gamma^2}{\sigma_\delta^2} \right] \int_0^\infty \frac{d\lambda\, \lambda^{1/2} \left[a_l \lambda + b_l \right]}{\left\{ \lambda^3 + a\lambda^2 + b\lambda + c \right\}^{3/2}}$$

$$\frac{1}{\tau_z} = \frac{\pi^2 \alpha^2 MN(\log)}{\gamma\tilde{\Gamma}} \left[\frac{\beta_z}{\varepsilon_z} \right] \int_0^\infty \frac{d\lambda\, \lambda^{1/2} \left[a_z \lambda + b_z \right]}{\left\{ \lambda^3 + a\lambda^2 + b\lambda + c \right\}^{3/2}}$$

with 9 coefficients in the integral to be determined *denominator coefficients (from determinant) for x, z, s*

	BM	**CM**	**New**
a	$\frac{\gamma^2 H_x}{\varepsilon_x}+\frac{\gamma^2}{\sigma_\delta^2}$	$\frac{\gamma^2 H_x}{\varepsilon_x}+\frac{\gamma^2}{\sigma_\delta^2}$ $+\frac{\beta_x}{\varepsilon_x}+\frac{\beta_z}{\varepsilon_z}$	$\left(\frac{\gamma^2}{\sigma_\delta^2}+\left(\frac{\beta_z}{\varepsilon_x}+\frac{\beta_x}{\varepsilon_z}\right)+\gamma^2\left(\frac{H_z}{\varepsilon_z}+\frac{H_x}{\varepsilon_x}\right)\right)$
b	$\left[\left(\frac{\beta_x}{\varepsilon_x}+\frac{\beta_z}{\varepsilon_z}\right)\left(\frac{\gamma^2 D_x^2}{\varepsilon_x\beta_x}+\frac{\gamma^2}{\sigma_\delta^2}\right)+\frac{\beta_x}{\varepsilon_x}\frac{\beta_z}{\varepsilon_z}\gamma^2\phi_x^2\right]$	$\left(\frac{\beta_x}{\varepsilon_x}+\frac{\beta_z}{\varepsilon_z}\right)\left(\frac{\gamma^2 D_x^2}{\varepsilon_x\beta_x}+\frac{\gamma^2}{\sigma_\delta^2}\right)$ $+\frac{\beta_x}{\varepsilon_x}\frac{\beta_z}{\varepsilon_z}\gamma^2\phi_x^2+\frac{\beta_x}{\varepsilon_x}\frac{\beta_z}{\varepsilon_z}$	$\left(\left(\frac{\beta_z}{\varepsilon_z}+\frac{\beta_x}{\varepsilon_x}\right)\left(\frac{\gamma^2}{\sigma_\delta^2}+\left(\frac{\gamma^2 D_z^2}{\beta_z\varepsilon_z}+\frac{\gamma^2 D_x^2}{\beta_x\varepsilon_x}\right)\right)+\gamma^2\frac{\beta_z\beta_x}{\varepsilon_x\varepsilon_z}\left(\varphi_x^2+\varphi_z^2\right)+\frac{\beta_z\beta_x}{\varepsilon_x\varepsilon_z}\right)$
c	$\frac{\beta_x}{\varepsilon_x}\frac{\beta_z}{\varepsilon_z}\left(\frac{\gamma^2 D_x^2}{\varepsilon_x\beta_x}+\frac{\gamma^2}{\sigma_\delta^2}\right)$	$\frac{\beta_x}{\varepsilon_x}\frac{\beta_z}{\varepsilon_z}\left(\frac{\gamma^2 D_x^2}{\varepsilon_x\beta_x}+\frac{\gamma^2}{\sigma_\delta^2}\right)$	$\left(\frac{\beta_z\beta_x}{\varepsilon_x\varepsilon_z}\left(\frac{\gamma^2}{\sigma_\delta^2}+\left(\frac{\gamma^2 D_z^2}{\beta_z\varepsilon_z}+\frac{\gamma^2 D_x^2}{\beta_x\varepsilon_x}\right)\right)\right)$

In the limit of zero vertical dispersion new coefficients reduce to CM ones

numerator coefficients for x

	BM	**CM**	**New**
$\mathbf{a_x}$	$2\frac{\gamma^2 H_x}{\varepsilon_x}+\frac{2\gamma^2}{\sigma_\delta^2}$	$\left(\frac{2\gamma^2 H_x}{\varepsilon_x}+\frac{2\gamma^2}{\sigma_\delta^2}-2\frac{\beta_x}{\varepsilon_x}-\frac{\beta_z}{\varepsilon_z}+\frac{\beta_x}{\gamma^2 H_x}\left(\boxed{6\frac{\beta_x}{\varepsilon_x}\gamma^2\phi_x^2}-\frac{\gamma^2}{\sigma_\delta^2}+2\frac{\beta_x}{\varepsilon_x}-\frac{\beta_z}{\varepsilon_z}\right)\right)$	$\left(2\gamma^2\left(\frac{H_z}{\varepsilon_z}+\frac{H_x}{\varepsilon_x}\right)-\frac{\beta_x H_z}{H_x\varepsilon_z}+2\frac{\gamma^2}{\sigma_\delta^2}-2\frac{\beta_x}{\varepsilon_x}-\frac{\beta_z}{\varepsilon_z}+\frac{\beta_x}{H_x\gamma^2}\left(\left(\frac{2\beta_x}{\varepsilon_x}-\frac{\beta_z}{\varepsilon_z}\right)-\frac{\gamma^2}{\sigma_\delta^2}\right)\right)$
$\mathbf{b_x}$	$\left(\frac{\beta_x}{\varepsilon_x}+\frac{\beta_z}{\varepsilon_z}\right)\left(\frac{\gamma^2 D_x^2}{\varepsilon_x\beta_x}+\frac{\gamma^2}{\sigma_\delta^2}\right)+\frac{\beta_x}{\varepsilon_x}\frac{\beta_z}{\varepsilon_z}\gamma^2\phi_x^2$	$\left(\frac{\beta_x}{\varepsilon_x}+\frac{\beta_z}{\varepsilon_z}\right)\left(\frac{\gamma^2 H_x}{\varepsilon_x}+\frac{\gamma^2}{\sigma_\delta^2}\right)-\frac{\beta_x^2}{\varepsilon_x^2}\gamma^2\phi_x^2+\left(\frac{\beta_x}{\varepsilon_x}-4\frac{\beta_z}{\varepsilon_z}\right)\frac{\beta_x}{\varepsilon_x}+\frac{\beta_x}{\gamma^2 H_x}\left(\frac{\gamma^2}{\sigma_\delta^2}\left(\frac{\beta_x}{\varepsilon_x}-2\frac{\beta_z}{\varepsilon_z}\right)\boxed{+6\frac{\beta_x}{\varepsilon_x}\frac{\beta_z}{\varepsilon_z}\gamma^2\phi_x^2}+\frac{\beta_x}{\varepsilon_x}\frac{\beta_z}{\varepsilon_z}-\gamma^2\frac{\beta_x^2\varphi_x^2}{\varepsilon_x^2}\right)$	$\left(\frac{\beta_z}{\varepsilon_z}+\frac{\beta_x}{\varepsilon_x}\right)\left(\frac{\gamma^2}{\sigma_\delta^2}+\frac{\gamma^2 H_x}{\varepsilon_x}\right)+\left(\frac{\beta_z}{\varepsilon_z}+\frac{\beta_x}{\varepsilon_x}\right)\gamma^2\frac{H_z}{\varepsilon_z}-\gamma^2\left(\frac{\beta_x^2}{\varepsilon_x^2}\varphi_x^2+\frac{\beta_z^2}{\varepsilon_z^2}\varphi_z^2\right)+\left(\frac{\beta_x}{\varepsilon_x}-4\frac{\beta_z}{\varepsilon_z}\right)\frac{\beta_x}{\varepsilon_x}+\frac{\beta_x H_z}{\varepsilon_z H_x}\left(\frac{\beta_x}{\varepsilon_x}-2\frac{\beta_z}{\varepsilon_z}\right)+\frac{\beta_x}{\gamma^2 H_x}\left(\frac{\gamma^2}{\sigma_\delta^2}\left(\frac{\beta_x}{\varepsilon_x}-2\frac{\beta_z}{\varepsilon_z}\right)+\frac{\beta_z\beta_x}{\varepsilon_z\varepsilon_x}+\gamma^2\left(2\frac{\beta_z^2\varphi_z^2}{\varepsilon_z^2}-\frac{\beta_x^2\varphi_x^2}{\varepsilon_x^2}\right)\right)$

In the limit of zero vertical dispersion the red terms are not found on the right

numerator coefficients for s

	BM	CM	New
a_1	$2\frac{\gamma^2 H_x}{\varepsilon_x}+\frac{2\gamma^2}{\sigma_\delta^2}$	$\left(2\frac{\gamma^2 H_x}{\varepsilon_x}+\frac{2\gamma^2}{\sigma_\delta^2}-\frac{\beta_x}{\varepsilon_x}-\frac{\beta_z}{\varepsilon_z}\right)$	$\left(2\frac{\gamma^2}{\sigma_\delta^2}-\frac{\beta_z}{\varepsilon_z}-\frac{\beta_x}{\varepsilon_x}+2\gamma^2\left(\frac{H_z}{\varepsilon_z}+\frac{H_x}{\varepsilon_x}\right)\right)$
b_1	$\left[\left(\frac{\beta_x}{\varepsilon_x}+\frac{\beta_z}{\varepsilon_z}\right)\left(\frac{\gamma^2 D_x^2}{\varepsilon_x\beta_x}+\frac{\gamma^2}{\sigma_\delta^2}\right)+\frac{\beta_x}{\varepsilon_x}\frac{\beta_z}{\varepsilon_z}\gamma^2\phi_x^2\right]$	$\left(\frac{\beta_x}{\varepsilon_x}+\frac{\beta_z}{\varepsilon_z}\right)\left(\frac{\gamma^2 D_x^2}{\varepsilon_x\beta_x}+\frac{\gamma^2}{\sigma_\delta^2}\right)+\frac{\beta_x}{\varepsilon_x}\frac{\beta_z}{\varepsilon_z}\gamma^2\phi_x^2-2\frac{\beta_x}{\varepsilon_x}\frac{\beta_z}{\varepsilon_z}$	$\left(\left(\frac{\beta_z}{\varepsilon_z}+\frac{\beta_x}{\varepsilon_x}\right)\frac{\gamma^2}{\sigma_\delta^2}-2\frac{\beta_x\beta_z}{\varepsilon_x\varepsilon_z}+\left(\frac{\beta_z}{\varepsilon_z}+\frac{\beta_x}{\varepsilon_x}\right)\gamma^2\left(\frac{H_z}{\varepsilon_z}+\frac{H_x}{\varepsilon_x}\right)-\gamma^2\left(\frac{\beta_x^2\phi_x^2}{\varepsilon_x^2}+\frac{\beta_z^2\phi_z^2}{\varepsilon_z^2}\right)\right)$

In the limit of zero vertical dispersion new coefficients reduce to CM ones

numerator coefficients for z

	BM	CM	New
$\mathbf{a_z}$	$-\frac{\gamma^2 H_x}{\varepsilon_x}-\frac{\gamma^2}{\sigma_\delta^2}$	$-\left(\frac{\gamma^2 H_x}{\varepsilon_x}+\frac{\gamma^2}{\sigma_\delta^2}+\frac{\beta_x}{\varepsilon_x}-2\frac{\beta_z}{\varepsilon_z}\right)$	$-\frac{\gamma^2}{\sigma_\delta^2}-\frac{H_x}{\varepsilon_x}\gamma^2+\left(\frac{2\beta_z}{\varepsilon_z}-\frac{\beta_x}{\varepsilon_x}\right)-\frac{\beta_x}{\varepsilon_x}\frac{\gamma^2 H_z}{\beta_z}-2\frac{\gamma^2 H_z}{\varepsilon_z}+2\gamma^4\frac{H_z}{\beta_z}\left(\frac{H_z}{\varepsilon_z}+\frac{H_z}{\varepsilon_z}\right)+2\frac{\gamma^4 H_z}{\beta_z\sigma_\delta^2}$
$\mathbf{b_z}$	$\left(\left(\frac{\beta_x}{\varepsilon_x}+\frac{\beta_z}{\varepsilon_z}\right)\left(\frac{\gamma^2 D_x^2}{\varepsilon_x\beta_x}+\frac{\gamma^2}{\sigma_\delta^2}\right)+\frac{\beta_x}{\varepsilon_x}\frac{\beta_z}{\varepsilon_z}\gamma^2\phi_x^2-3\frac{\beta_x}{\varepsilon_x}\left(\frac{\gamma^2 D_x^2}{\varepsilon_x\beta_x}+\frac{\gamma^2}{\sigma_\delta^2}\right)\right)$	$\left(\left(\frac{\beta_x}{\varepsilon_x}+\frac{\beta_z}{\varepsilon_z}\right)\left(\frac{\gamma^2 D_x^2}{\varepsilon_x\beta_x}+\frac{\gamma^2}{\sigma_\delta^2}\right)+\frac{\beta_x}{\varepsilon_x}\frac{\beta_z}{\varepsilon_z}\gamma^2\phi_x^2+\frac{\beta_x}{\varepsilon_x}\frac{\beta_z}{\varepsilon_z}-3\frac{\beta_x}{\varepsilon_x}\left(\frac{\gamma^2 D_x^2}{\varepsilon_x\beta_x}+\frac{\gamma^2}{\sigma_\delta^2}\right)\right)$	$\frac{\gamma^2}{\sigma_\delta^2}\left(\frac{\beta_z}{\varepsilon_z}-2\frac{\beta_x}{\varepsilon_x}\right)+\frac{\beta_z\beta_x}{\varepsilon_z\varepsilon_x}+\frac{\gamma^2 H_x}{\varepsilon_x}\left(\frac{\beta_z}{\varepsilon_z}-2\frac{\beta_x}{\varepsilon_x}\right)+\gamma^2\left(2\frac{\beta_x^2\varphi_x^2}{\varepsilon_x^2}-\frac{\beta_z^2\varphi_z^2}{\varepsilon_z^2}\right)+\left(\frac{\beta_z}{\varepsilon_z}-4\frac{\beta_x}{\varepsilon_x}\right)\frac{\gamma^2 H_z}{\varepsilon_z}+\left(\frac{\beta_z}{\varepsilon_z}+\frac{\beta_x}{\varepsilon_x}\right)\gamma^4\frac{H_z H_x}{\beta_z\varepsilon_x}-\gamma^4\frac{H_z}{\beta_z}\left(\frac{\beta_z^2}{\varepsilon_z^2}\varphi_z^2+\frac{\beta_x^2}{\varepsilon_x^2}\varphi_x^2\right)+\left(\frac{\beta_x}{\varepsilon_x}+\frac{\beta_z}{\varepsilon_z}\right)\left(\frac{\gamma^4 H_z^2}{\beta_z\sigma_\delta^2}+\frac{\gamma^4 H_z}{\beta_z\varepsilon_z}\right)$

In the limit of zero vertical dispersion new coefficients reduce to CM ones

check of effect of dropping the two terms from CM formula

Present MADX – protons at LHC injection
(Weighted) average lifetimes (sec):
Longitudinal= 3.800737E+04
Horizontal = 1.323025E+05
Vertical = -1.799992E+07
New MADX version - also protons at injection
(Weighted) average lifetimes (sec):
Longitudinal= 3.800737E+04
Horizontal = 1.320398E+05
Vertical = -1.799992E+07

Present MADX version - protons 5GeV/c
(Weighted) average lifetimes (sec):
Longitudinal= -3.766200E-01
Horizontal = 1.567772E-02
Vertical = 1.430763E-02
New MADX version - also protons 5GeV/c
Longitudinal= -3.766200E-01
Horizontal = 1.525223E-02
Vertical = 1.430763E-02

conclusions:

- a new more general formula for IBS growth rates was derived, which includes vertical dispersion
- this solution reduces to the CM solution for the vertical and longitudinal growth rate if D_z=0; but it is different for the horizontal growth rate
- close inspection of MAD-X code shows that it actually uses the CM solution and not the original BM formulae, as had been thought!
- for LHC at injection the difference in the *x* growth rate due to the 2 different terms is less than 0.1%
- new formulae will soon be implemented in MAD-X