resistive-wall wake field for two beams propagating in opposite directions

considerations:

- longitudinal wake field, $\sim E_s$, changes sign at long distances it becomes decelerating instead of accelerating
- total effect on particle in opposite beam must be obtained by integrating over z instead of keeping z constant; contribution from short-range wake component likely significant (theory of Bane and Sands)
- dipole longitudinal impedance also changes sign
- transverse electric force stays the same, but magnetic force inverts sign (which part dominates the wake?)
- Panofsky-Wenzel theorem no longer holds

Panofsky-Wenzel theorem for 1 beam

$$\frac{\partial}{\partial z}\left(E_r + \left(\vec{v}\times\vec{B}\right)_r\right) = \frac{\partial E_s}{\partial r}$$

proof:

$$\frac{\partial}{\partial z}\left(-\left(\frac{\partial\phi}{\partial r}\right) + v_z\frac{\partial A_s}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\frac{\partial\phi}{\partial s} - \frac{\partial A_s}{\partial t}\right)$$

using $\frac{\partial}{\partial z} = \frac{\partial}{\partial s} = -\frac{\partial}{v_z\partial t}$

with two beams

$$A_s,\phi = A_s,\phi\left(z_1 \equiv s_1 - v_z t = -s_2 - v_z t = -z_2 - 2v_z t\right)$$

and

$$\frac{\partial}{\partial z_2}\left(-\left(\frac{\partial\phi}{\partial r}\right) + v_z\frac{\partial A_s}{\partial r}\right) \neq \frac{\partial}{\partial r}\left(-\frac{\partial\phi}{\partial s_2} - \frac{\partial A_s}{\partial t}\right)$$

since

$$\frac{\partial}{\partial z_2} = \frac{\partial}{\partial s_2} = +\frac{1}{2}\frac{\partial}{v_z\partial t}$$

$$z_1 \equiv s_1 - v_z t = -z_2 - 2v_z t$$

$$s_1 = -s_2$$

$$z_2 \equiv s_2 - v_z t$$

first, consider the classical regime of the res.-wall wake (A. Chao's book)

$$b/\chi \gg |z| \gg \chi^{1/3} b \quad \text{with} \quad \chi \equiv \frac{c}{4\pi\sigma b}$$

$$W_\perp \propto -\left(E_r - cB_\theta\right)$$

this term cancels in Lorentz force

conventional wake field:

$$cB_\theta = E_r - \frac{2qa}{\pi b^3}\sqrt{\frac{c}{\sigma}}\frac{1}{|z|^{1/2}}$$

$$W_\perp = -\frac{2}{\pi b^3}\sqrt{\frac{c}{\sigma}}\frac{1}{|z|^{1/2}}$$

same long-range wake, with opposite sign

no longer cancellation!

2-beam wake field:

$$cB_\theta = -E_r + \frac{2qa}{\pi b^3}\sqrt{\frac{c}{\sigma}}\frac{1}{|z|^{1/2}}$$

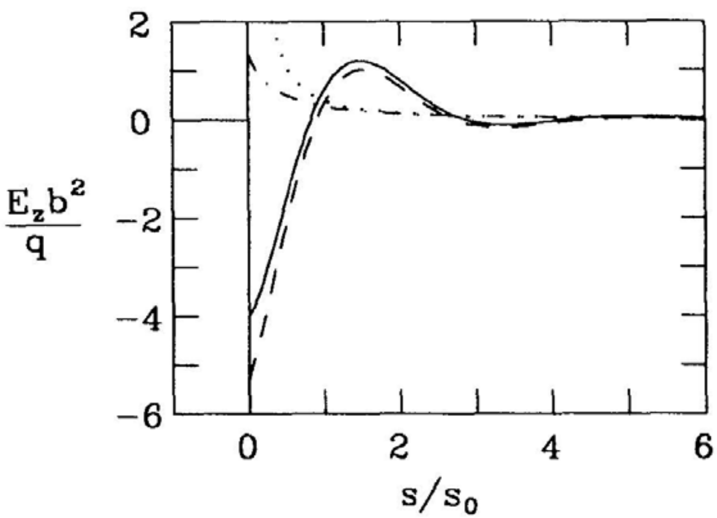
z integral diverges!

opposite sign

$$W_\perp^{1\to 2} = \left[\frac{3\left(r^2+b^2\right)}{4\pi b^3}\sqrt{\frac{c}{\sigma}}\frac{1}{|z|^{5/2}} + \frac{2}{\pi b^3}\sqrt{\frac{c}{\sigma}}\frac{1}{|z|^{1/2}}\right]$$

to get a finite value for the integral, need we derive & include short-range part?

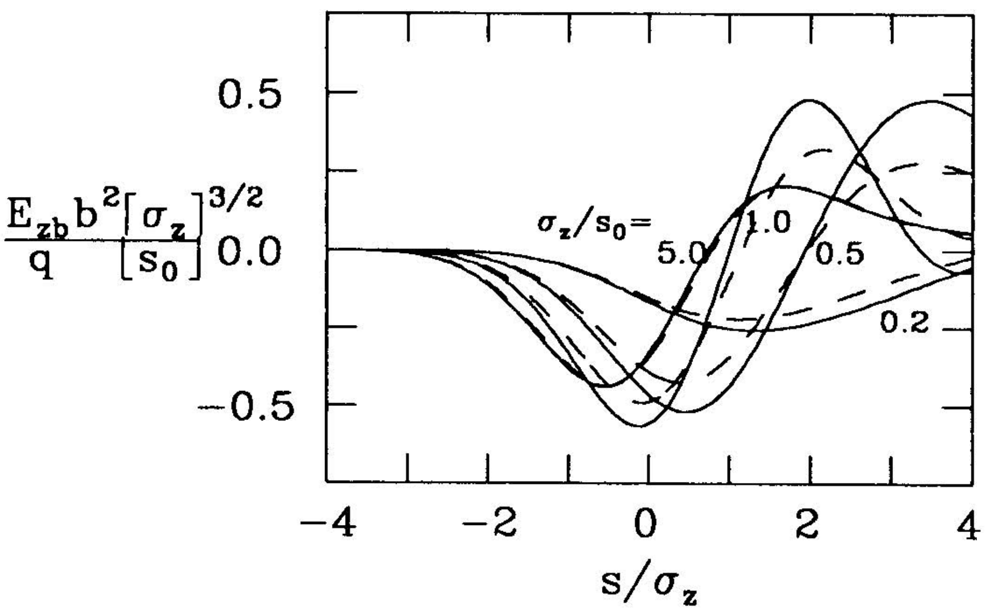
single-beam short-range resistive-wall wake field (K. Bane & M. Sands, SLACL-PUB-95-7074)



short range Green-function wake field (solid), also shown oscillator component (dashes), diffusion term (dot-dashes), and long-range solution (dots)

$$s_0 \equiv \left(\frac{b^2}{\sigma} \frac{2}{Z_0} \right)^{1/3} \approx 0.5\,\mathrm{mm}$$

longitudinal wake field of a Gaussian bunch for a dc conductivity (dashes) and ac conductivity with Γ=0.4; curves are for several values of σ_z/s_0, if σ_z/s_0 >5, long-range wake can be used



we need to find the short-range behavior of the radial electric and azimuthal magnetic fields; return to derivation in Alex' book (actually this may go back to Morton, Neil and Sessler, 1966); and solve the problem by Fourier transform

$$E_r(z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikz} \tilde{E}_r(k)$$

$$B_\theta(z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikz} \tilde{B}_\theta(k)$$

$\longrightarrow$

$$\tilde{E}_r = -\frac{ikA}{4} r^2 + \frac{1}{2}\left(-\frac{iA}{k} + B\right)$$

$$\tilde{B}_\theta = -\frac{ikA}{4} r^2 + \frac{1}{2}\left(\frac{iA}{k} + B\right)$$

2-beam wake: $$\tilde{E}_r + \tilde{B}_\theta = -\frac{ikA}{2} r^2 + B \approx B$$

1-beam wake: $$\tilde{E}_r - \tilde{B}_\theta = -\frac{iA}{k}$$

coefficients *A* and *B* from field matching at chamber surface

approximation: $|\lambda| >> \frac{1}{b}, \frac{1}{t}$

$$A = \frac{4qa}{b^3\left(\frac{ikb}{2} - \frac{\lambda}{k} - \frac{i}{kb}\right)}$$

Bane & Sands neglected this last term

$$\lambda^2 = \frac{4\pi\sigma ik}{c}$$

take root in upper complex plane

Chao's long-range approximation keeps only this!

$$B = -\frac{\lambda}{k} bA = -\frac{4qa}{b^2\left(\frac{ik^2 b}{2\lambda} - 1 - \frac{i}{b\lambda}\right)}$$

(1) extremely short-range wake

$$k^2 >> \frac{2}{b^2}$$

above pipe cutoff

$$B = -\frac{\lambda}{k} bA = -\frac{4qa}{b^2\left(\frac{ik^2 b}{2\lambda} - 1 - \frac{i}{b\lambda}\right)}$$

drop the last term as Bane and Sands did

$$\rightarrow \quad B \approx -\frac{8qa}{b^3} X(1-i) \frac{1}{k^{3/2} - Y(1-i)}$$

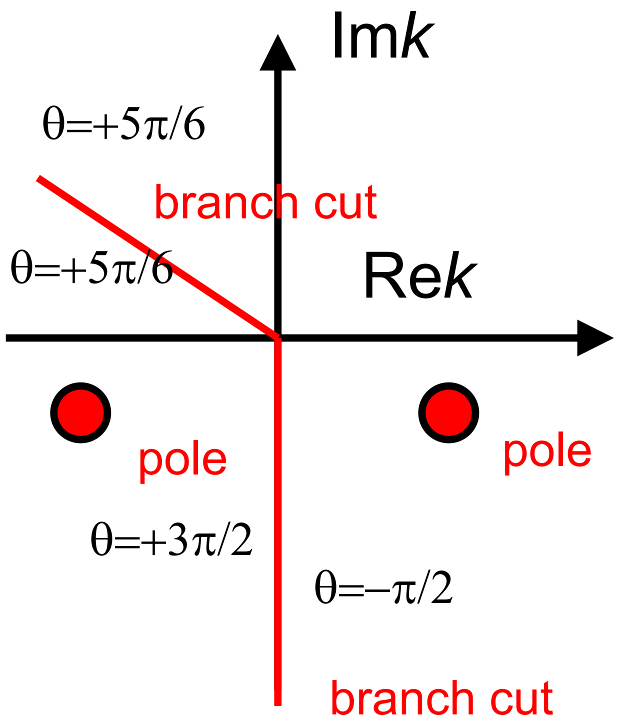
with $X \equiv \sqrt{\frac{2\pi\sigma}{c}}, Y \equiv \frac{2}{b} X$

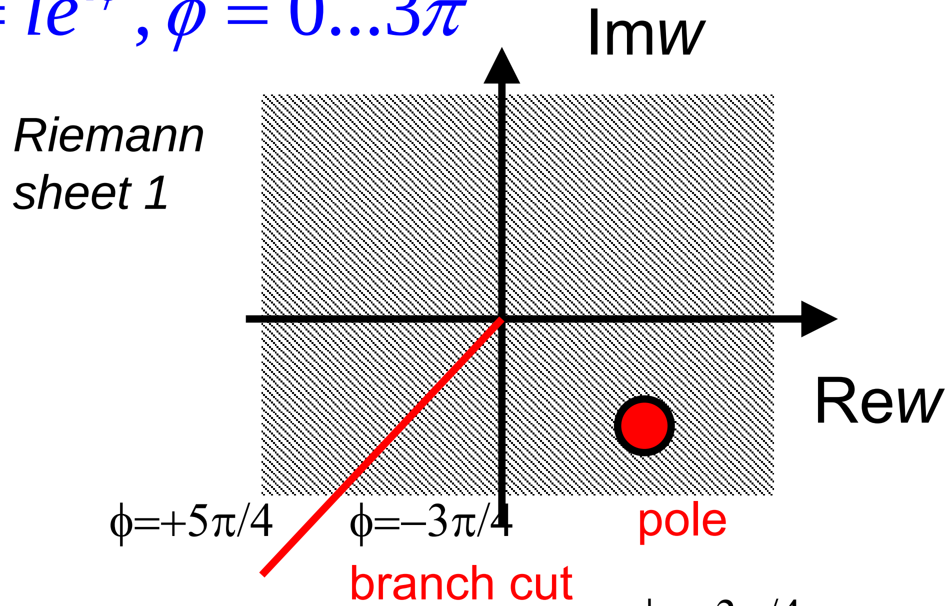
or

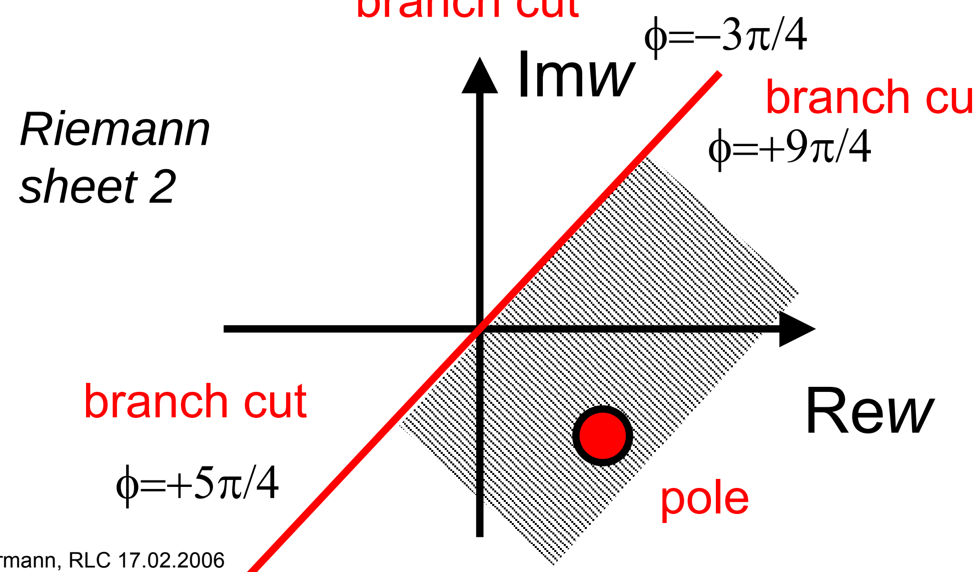
$$B \approx D \frac{1}{k^{3/2} - Y(1-i)}$$

with $D \equiv -\frac{8qa}{b^3} X(1-i)$

$$w = k^{3/2} \longrightarrow \begin{array}{l} k = re^{i\theta}, \theta = 0 \ldots 2\pi \\ w = le^{i\phi}, \phi = 0 \ldots 3\pi \end{array}$$







poles:

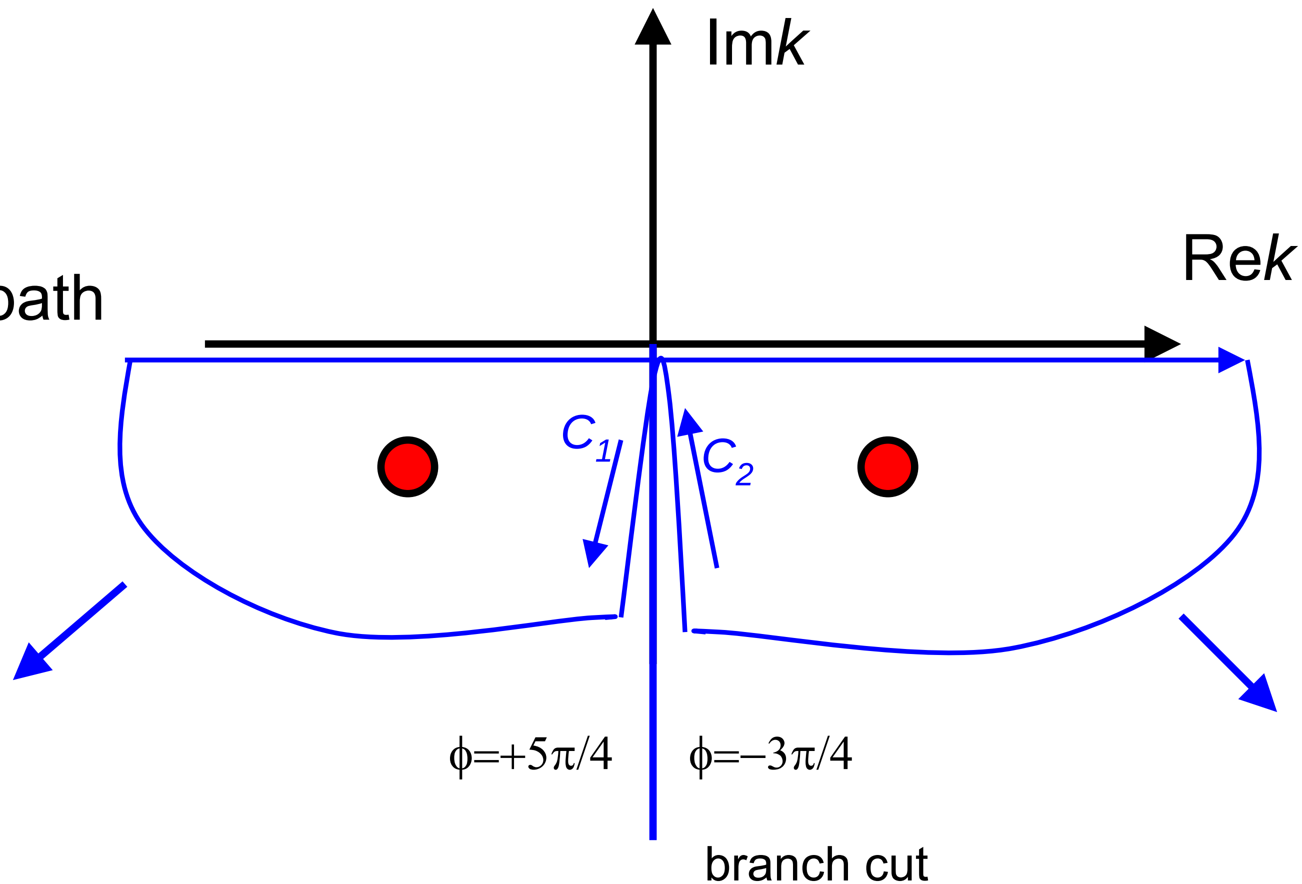
$$w = Y(1-i)$$

$$\longleftrightarrow \quad k = Y^{2/3} 2^{1/3} \begin{cases} e^{-i\pi/6} \\ e^{i7\pi/6} \end{cases}$$

poles:

$$k = Y^{2/3}\,2^{1/3}\begin{cases} e^{-i\pi/6} \\ e^{i7\pi/6} \end{cases}$$

integration path for $z<0$



$$F(z, r=0)/e = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikz} B$$

$$= -i \sum_{\text{poles}} \text{res}\left(e^{ikz} B\right) - \int_{\substack{\text{branch cut}\\ C_1 + C_2}} \frac{1}{2\pi} \left(\mathrm{e}^{\mathrm{ikz}} B\right) dk$$

to get the residues we need the expansion $\frac{1}{z^{3/2} - a^{3/2}} = \frac{2}{3\sqrt{a}(z-a)} - \ldots$

$$-i \sum_{\text{poles}} \text{res}\left(e^{ikz} B\right)$$

$$= -iD\left[e^{izY^{2/3} 2^{1/3} e^{-i\pi/6}} \frac{2}{3\sqrt{Y^{2/3} 2^{1/3} e^{-i\pi/6}}} + e^{izY^{2/3} 2^{1/3} e^{i7\pi/6}} \frac{2}{3\sqrt{Y^{2/3} 2^{1/3} e^{i7\pi/6}}} \right]$$

$$= -iD \frac{2}{3Y^{1/3} 2^{1/6}} \left[e^{izY^{2/3} 2^{1/3} e^{-i\pi/6} + i\pi/12} + e^{izY^{2/3} 2^{1/3} e^{i7\pi/6} - i7\pi/12} \right]$$

$$-i\sum_{\text{poles}} \mathrm{res}\left(e^{ikz}B\right)$$

$$= -iD\,\frac{2}{3Y^{1/3}2^{1/6}}\left[e^{izY^{2/3}2^{1/3}\left(\sqrt{3}/2-i/2\right)+i\pi/12}+e^{izY^{2/3}2^{1/3}\left(-\sqrt{3}/2-i/2\right)-i7\pi/12}\right]$$

$$= -iD\,\frac{2}{3Y^{1/3}2^{1/6}}\,e^{zY^{2/3}2^{-2/3}}\left[e^{izY^{2/3}2^{1/3}\left(\sqrt{3}/2\right)+i\pi/12}+e^{izY^{2/3}2^{1/3}\left(-\sqrt{3}/2\right)-i7\pi/12}\right]$$

$$= \frac{8qa}{b^3}X\,\frac{2^{7/3}}{3Y^{1/3}}\,e^{zY^{2/3}2^{-2/3}}\left[\frac{e^{izY^{2/3}2^{1/3}\sqrt{3}/2+i\pi/3}+e^{-izY^{2/3}2^{1/3}\sqrt{3}/2-i\pi/3}}{2}\right]$$

$$= \frac{8qa}{b^3}X\,\frac{2^{7/3}}{3Y^{1/3}}\,e^{zY^{2/3}2^{-2/3}}\cos[\,zY^{2/3}2^{-2/3}\sqrt{3}+\pi/3]$$

$$-i\sum_{\text{poles}} \mathrm{res}\left(e^{ikz}B\right)=\frac{32qa}{3b^3}X^{2/3}b^{1/3}e^{zb^{-2/3}X^{2/3}}\cos[zb^{-2/3}X^{2/3}\sqrt{3}+\pi/3]$$

contribution from the two poles

$$\int_{\substack{\text{branch cut}\\ C_1+C_2}} \frac{1}{2\pi}\left(\mathrm{e}^{\mathrm{ikz}}B\right)dk$$

$$= -iD\int_0^\infty \frac{du}{2\pi}e^{uz}\left(\frac{1}{e^{i5\pi/4}u^{3/2}-Y(1-i)}-\frac{1}{e^{-i3\pi/4}u^{3/2}-Y(1-i)}\right)$$

$$= -iD\int_0^\infty \frac{du}{2\pi}e^{uz}\left(\frac{\left(e^{-i3\pi/4}-e^{i5\pi/4}\right)u^{3/2}}{iu^3-Yu^{3/2}\left(e^{i5\pi/4}+e^{-i3\pi/4}\right)-2iY^2}\right)$$

$$= \frac{8qa}{b^3}X\sqrt{2}\int_0^\infty \frac{du}{2\pi}e^{uz}\left(\frac{\left(e^{-i4\pi/4}-e^{i4\pi/4}\right)u^{3/2}}{u^3+e^{i\pi/2}Yu^{3/2}\left(e^{i5\pi/4}+e^{-i3\pi/4}\right)-2Y^2}\right)$$

$$= 0$$

I find 0 contribution from the branch cut short-range 2-beam Green-function wake field in Gaussian units

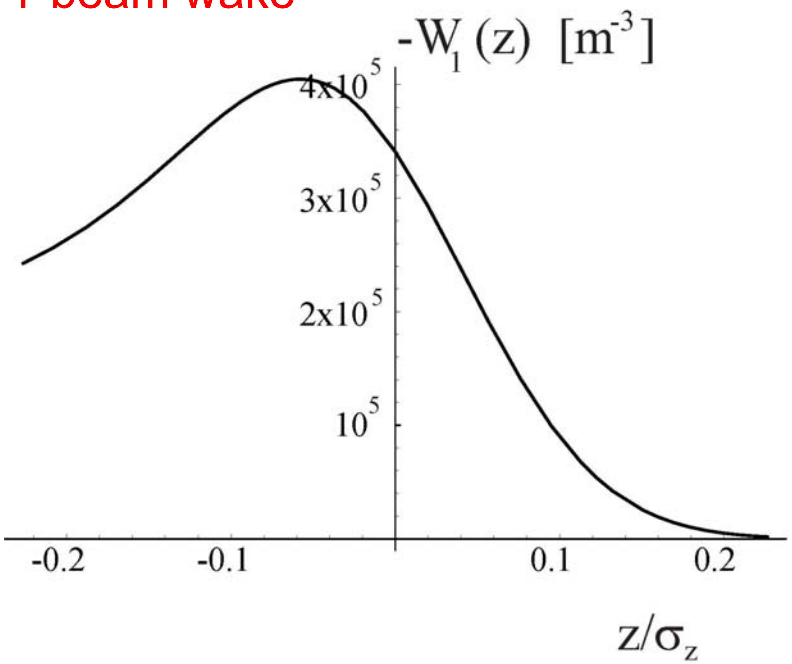
$$W_{1\to 2}(z) = -\frac{32}{3b^3} X^{2/3} b^{1/3} e^{zX^{2/3} b^{-2/3}} \cos\left[zX^{2/3} b^{-2/3}\sqrt{3} + \frac{\pi}{3} \right]$$

$$= -\frac{32}{3b^3}\left(\frac{2\pi\sigma}{c}\right)^{1/3} b^{1/3} e^{z(2\pi\sigma/c)^{1/3} b^{-2/3}} \cos\left[\frac{z\sqrt{3}(2\pi\sigma/c)^{1/3}}{b^{2/3}} + \frac{\pi}{3} \right]$$

conventional 1-beam long-range Green-function wake field

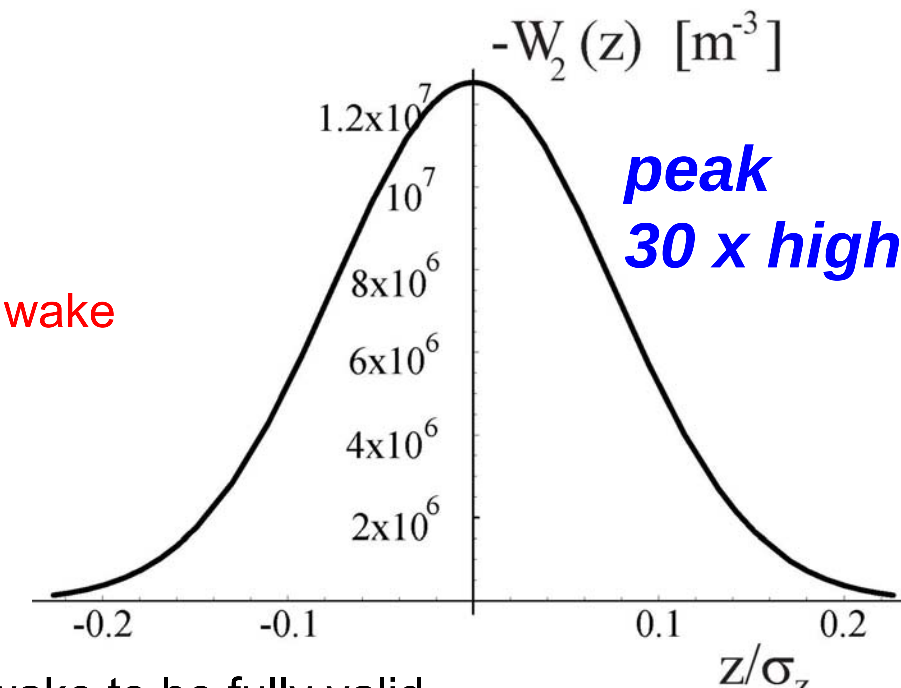
$$W_{\perp}(z) = -\frac{2}{\pi b^3}\sqrt{\frac{c}{\sigma}}\frac{1}{|z|^{1/2}}$$

1-beam wake



wake generated
by Gaussian bunch
for
b=1.5 mm
ρ=10 μΩm
σ_z=7.55 cm

2-beam wake



peak
30 x higher!

short-range
resistive-wall
wake

bunch length here is too long for the wake to be fully valid

remarks

- this wake could enhance the effect of long-range collisions
- formula is correct for CLIC bunch lengths

problems

- consistency with long-range result?
- in particular, long-range term should come from branch-cut integral which seems to be zero

(2) longer-range wake

$$B = -\frac{\lambda}{k} bA = -\frac{4qa}{b^2\left(\frac{ik^2 b}{2\lambda} - 1 - \frac{i}{b\lambda}\right)}$$

drop the first term which seems small for collimator parameters

$$k^2 << \frac{2}{b^2}$$

below pipe cutoff

for example:

1) *b*=1.5 mm, ρ = 10 μΩm, *f*=100 MHz

→ $k^2 b/(2\lambda)$ ~4x10^{-7} m^{-2}, $1/(\lambda\, b)$~0.08 m^{-2}

2) *b*=1.5 mm, ρ = 10 μΩm, *f*=10 GHz

→ $k^2 b/(2\lambda)$ ~0.01 m^{-2}, $1/(\lambda\, b)$~0.002 m^{-2}

$$B \approx \frac{4qa}{b^2} \frac{1}{1 + \frac{i}{b\sqrt{\frac{2\pi\sigma}{c}}(1+i)k^{1/2}}}$$

or

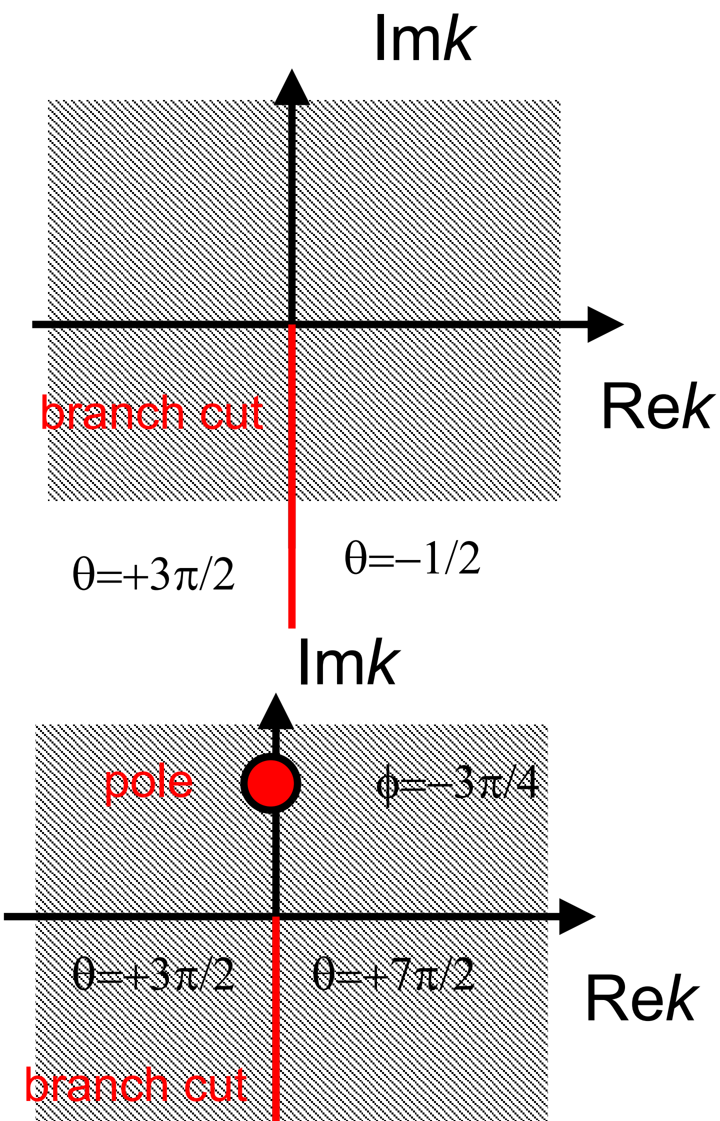
$$B \approx \frac{4qa}{b^2} \frac{k^{1/2}}{k^{1/2} + \frac{1+i}{2bX}}$$

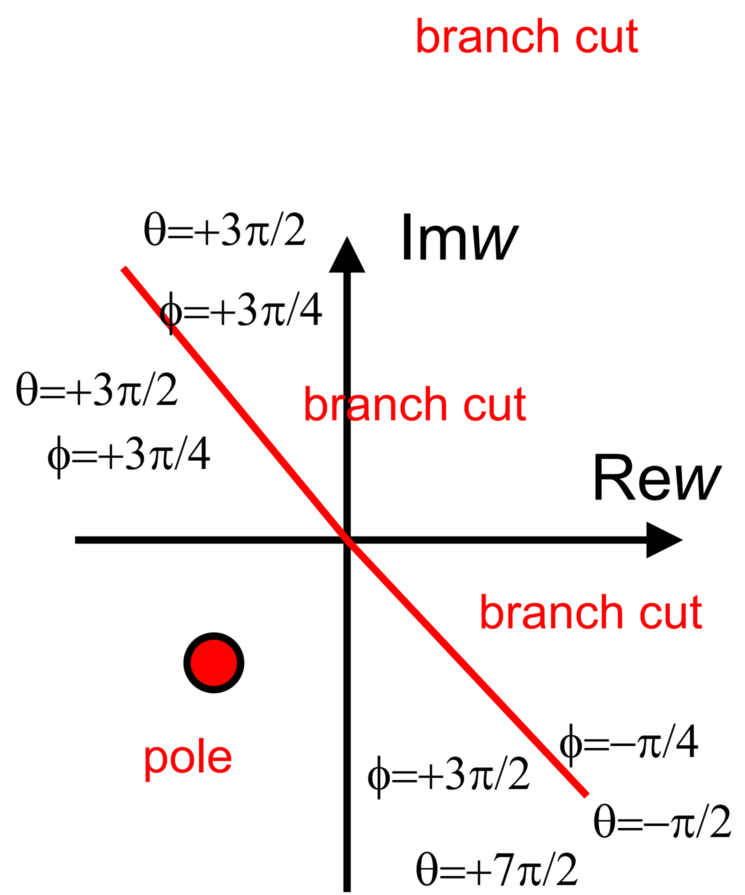
$$w = k^{1/2} \longrightarrow \begin{array}{l} k = re^{i\theta},\ \theta = 0\ldots4\pi \\ w = le^{i\phi},\ \phi = 0\ldots2\pi \end{array}$$

Riemann sheet 1

no pole in the 1st Riemann sheet!

Riemann sheet 2





$$\int_{\substack{\text{branch cut}\\ C_1+C_2}} \frac{1}{2\pi}\left(\mathrm{e}^{\mathrm{ikz}} B\right) dk$$

$$= -i\,\frac{4qa}{b^2}\int_0^\infty \frac{du}{2\pi} e^{uz}\left(\frac{u^{1/2} e^{i3\pi/4}}{e^{i3\pi/4} u^{1/2} + \frac{(1+i)}{2bX}} - \frac{u^{1/2} e^{-i\pi/4}}{e^{-i\pi/4} u^{1/2} + \frac{(1+i)}{2bX}} \right)$$

$$= \frac{4qa}{b^2}\int_0^\infty \frac{du}{2\pi} e^{uz}\left(\frac{u^{1/2}\,\frac{\sqrt{2}}{bX}}{u + \frac{1}{2b^2X^2}} \right)$$

$$= \frac{4\sqrt{2}qa}{b^3 X}\,\frac{1}{2\pi}\left[\frac{\sqrt{\pi}}{\sqrt{|z|}} + e^{z/\left(2b^2X^2\right)}\pi\,\frac{1}{bX\sqrt{2}}\left(\mathrm{Erf}\left(\frac{\sqrt{|z|}}{\sqrt{2}bX} \right) - 1 \right) \right]$$

short-range 2-beam Green-function wake field in Gaussian units

$$W_{1\to 2}(z)=\frac{2}{b^3}\left[\frac{1}{\pi}\frac{\sqrt{c/\sigma}}{\sqrt{|z|}}+e^{z/\left(4b^2\pi\sigma/c\right)}\frac{c}{2\pi\sigma}\frac{1}{b}\left(\mathrm{Erf}\left(\frac{\sqrt{|z|}}{2b\sqrt{\pi\sigma/c}}\right)-1\right)\right]$$

conventional 1-beam long-range Green-function wake field

$$W_{\perp}(z)=-\frac{2}{\pi b^3}\sqrt{\frac{c}{\sigma}}\frac{1}{|z|^{1/2}}$$

longer-range 2-beam wake field is essentially equal and opposite to the classical one (as expected from slide 4)

there is however a difference in the calculation of the total wake effect:

$$W_{tot,1\to 2}=\int_{s_{\mathrm{encounter}}}^{L}W(z)dz$$

versus

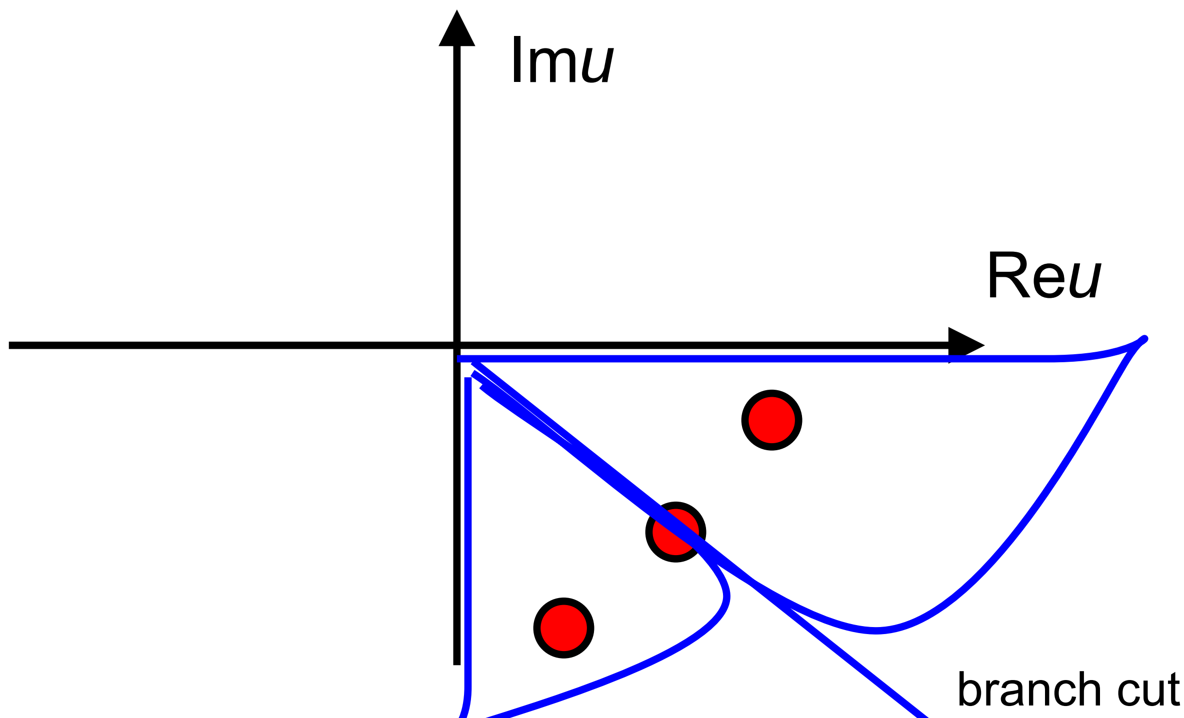
$$W_{tot,\perp}=W_{\perp}(z)L$$

this is the wake to use for the LHC

introduce new variable $u \equiv k^{1/2}$ *is this permitted?*

poles are

$$u_{pole} = (2Y)^{1/6} \begin{cases} e^{-i\pi/12} \\ e^{-i3\pi/12} \\ e^{-i5\pi/12} \end{cases}$$



contribution from residues

$$F = \int_{-i\infty}^{\infty} \frac{du}{\pi} u e^{iu^2 z} \left(-\frac{8eqa}{b^2} X(1-i)\right)\left[\frac{1}{u-u_{p,1}}\frac{1}{u-u_{p,2}}\frac{1}{u-u_{p,3}}\right]$$

$$= \left(\frac{16eqa}{b^2}X(1-i)\right)\left[\frac{1}{u_{p,1}-u_{p,2}}\frac{1}{u_{p,1}-u_{p,3}} u_{p,1} e^{-iu_{p,1}^2 z} + \frac{1}{u_{p,3}-u_{p,2}}\frac{1}{u_{p,3}-u_{p,1}} u_{p,3}e^{-iu_{p,3}^2 z}\right]$$

$$+ \int_0^\infty \frac{dr}{\pi} r e^{-i\pi/2} e^{r^2 z}\left(-\frac{8eqa}{b^2}X(1-i)\right)\left[\frac{1}{re^{-i\pi/4}-u_{p,1}}\frac{1}{re^{-i\pi/4}-u_{p,2}}\left(\frac{1}{-re^{-i\pi/4}-u_{p,3}} - \frac{1}{re^{-i\pi/4}-u_{p,3}}\right)\right]$$

$$\left[\frac{2re^{-i\pi/4}}{ir^2+u_{p,3}^2}\right]$$

contribution from branch cut

$$\left[\frac{1}{u_{p,1}-u_{p,2}}\frac{1}{u_{p,1}-u_{p,3}} u_{p,1} e^{-iu_{p,1}^2 z} + \frac{1}{u_{p,3}-u_{p,2}}\frac{1}{u_{p,3}-u_{p,1}} u_{p,3}e^{-iu_{p,3}^2 z}\right]$$

$$= \left(\frac{e^{-i\pi 5/12} e^{-(2Y)^{1/3}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)z}\left(e^{i\pi/6}+e^{i(2Y)^{1/3}\sqrt{3} z}\right)}{(2Y)^{1/6}\left(-1+e^{-i\pi/3}\right)\left(-1+e^{-i\pi/6}\right)}\right)$$

$$B \approx -\left(\frac{8qa}{ib^2}\right)\left\{\frac{2\pi\sigma}{kc} - \frac{2\pi\sigma}{c\left(k + \frac{c}{2\pi\sigma}\right)}\right\}$$

$$F(z) = \left(\frac{8qa}{b^2}\right)\left(1 + e^{-i\frac{2\pi\sigma}{c}z}\right)$$

$$A=\frac{4qa}{b^3\left(\frac{ikb}{2}-\frac{\lambda}{k}-\frac{i}{kb}\right)}=\frac{4qac}{b^4 2\pi\sigma}\frac{1}{\left(\frac{2i\kappa b^2}{\delta^2}-\frac{i+\mathrm{sgn}(\kappa)}{|\kappa|^{1/2}}-\frac{i}{\kappa}\right)}=\frac{8qa}{b^2\delta}\frac{1}{\left(\frac{2i\kappa b^2}{\delta^2}-\frac{1+i}{\kappa^{1/2}}-\frac{i}{\kappa}\right)}$$

$$\kappa\equiv\frac{2\pi\sigma kb^2}{c}$$

$$\delta\equiv\frac{4\pi\sigma b^2}{c}$$

analytical extension, branch cut on negative imaginary κ axis

Bane & Sands neglected this last term when they derived the wake field

→ calculation of residues and integration over branch cuts

$$E_r(z)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikz}\tilde{E}_r(k)$$

$$=-\int_{-i\infty}^{+\infty}\frac{u^3du}{2\pi}\frac{c}{2\pi\sigma b}\frac{\delta^{4/3}}{b^{4/3}}e^{iu^2}\left(\frac{2^{1/3}}{b^{10/3}\delta^{2/3}(1+i)}\right)8qa\frac{\sqrt{2}i}{\left(u^3-\frac{(1+i)}{\sqrt{2}i}\right)}e^{i\frac{\delta^{2/3}}{2^{1/6}b^{5/3}}u(1+i)(r-b)}$$

we could also start from the general solution of the electro-magnetic fields derived by Oide and myself in PRST-AB 044201

approximation: $|\lambda| >> \frac{1}{b}, \frac{1}{t}$

$$E_r = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \lim_{k_r \to 0} \left(-\frac{\partial}{\partial r} \left[c\left(p_s I_1(k_r r) + q_s K_1(k_r r) \right) \right] + i\omega p_+ \left(I_2(k_r r) - I_0(k_r r) \right) \right) e^{ikz}$$

$$E_r = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \lim_{k_r \to 0} \left(\begin{array}{l} -\left[\left(p_s \frac{ck_r}{2} \{ I_0(k_r r) + I_2(k_r r) \} - q_s \frac{ck_r}{2} \{ K_0(k_r r) + K_2(k_r r) \} \right) \right] \\ + i\omega p_+ \left(I_2(k_r r) - I_0(k_r r) \right) \end{array} \right) e^{ikz}$$